ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)}\ge 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \cot\frac{A}{2}\le\cot\frac{B}{2}\le\cot\frac{C}{2}$$

$$\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)}\overset{Chebyshev}{\ge}\frac{\frac{1}{2}(b^3+c^3)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{\frac{a}{2R}.2R(b^2+c^2)}\overset{Chebyshev}{\ge}$$

$$\ge\frac{\frac{1}{4}(b+c)(b^2+c^2)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{\frac{a}{2R}.2R(b^2+c^2)}=\frac{\frac{1}{4}\left((b+c)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)\right)}{a}\overset{AM-GM}{\ge}$$

$$\ge\frac{\sqrt{bc}}{a}\sqrt{\cot\frac{C}{2}\cot\frac{B}{2}}\quad(1)$$

$$\sum\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)}\overset{(1)}{\ge}\sum\frac{\sqrt{bc}}{a}\sqrt{\cot\frac{C}{2}\cot\frac{B}{2}}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\prod\cot\frac{A}{2}}=3\sqrt[3]{\frac{s}{r}}\overset{Mitrinovic}{\ge}3(3\sqrt{3})^{\frac{1}{3}}=3\sqrt{3}$$

$$Equality\ holds\ for\ \ a=b=c$$