ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a(b+c)}{9R^2-a^2}+\frac{b(c+a)}{9R^2-b^2}+\frac{c(a+b)}{9R^2-c^2}<4$$

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Solution by Tapas Das-India

$$We\ know\ that\ in\ \Delta ABC:$$

$$b+c>a\ or\ 2(b+c)>a+b+c\ or\ 2(b+c)>2s\ or\ b+c>s$$

$$similarly, (a+b)>s\ and\ (c+a)>s$$

$$9R^2-a^2\overset{Leibniz}{\geq}a^2+b^2+c^2-a^2=b^2+c^2\overset{CBS}{\geq}\frac{(b+c)^2}{2}\quad(1)$$

$$Similarly:\ \ 9R^2-b^2\geq\frac{(a+c)^2}{2}\quad(2),9R^2-c^2\geq\frac{(a+b)^2}{2}\quad(3)$$

$$\frac{a(b+c)}{9R^2-a^2}+\frac{b(c+a)}{9R^2-b^2}+\frac{c(a+b)}{9R^2-c^2}\overset{(1),(2)\&(3)}{\leq}\frac{a(b+c)}{\frac{(b+c)^2}{2}}+\frac{b(c+a)}{\frac{(a+c)^2}{2}}+\frac{c(a+b)}{\frac{(a+b)^2}{2}}=$$

$$=\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}<\frac{2a}{s}+\frac{2b}{s}+\frac{2c}{s}=\frac{2(a+b+c)}{s}=2\cdot\frac{2s}{s}=4$$

Equality holds for: $a=b=c$.