ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)^3}{\left(\sin^2 A+\cos^2\frac{A}{2}\right)^2}+\frac{\left(\sec^2\frac{B}{2}+\csc^2\frac{B}{2}\right)^3}{\left(\sin^2 B+\cos^2\frac{B}{2}\right)^2}+\frac{\left(\sec^2\frac{C}{2}+\csc^2\frac{C}{2}\right)^3}{\left(\sin^2 C+\cos^2\frac{C}{2}\right)^2}\ge\frac{16384}{81}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)=\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\sin^2\frac{A}{2}}=\frac{\left(\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right)}{\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=$$

$$=\frac{1}{\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=\frac{4}{4\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=\frac{4}{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)^2}=\frac{4}{\sin^2 A}\ (1)$$

$$\sum\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)\overset{(1)}{=}4\sum\frac{1}{\sin^2 A}\overset{Bergstom}{\ge}4\frac{(1+1+1)^2}{\sum\sin^2 A}=$$

$$=\frac{36}{\sum\frac{a^2}{4R^2}}=\frac{144R^2}{\sum a^2}\overset{Leibniz}{\ge}\frac{144R^2}{9R^2}=16\ (2)$$

$$\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}\overset{Euler}{\le}2+\frac{1}{4}=\frac{9}{4}\ (3)\ and$$

$$\sum\sin^2 A=\frac{a^2+b^2+c^2}{4R^2}\overset{Leibniz}{\le}\frac{9R^2}{4R^2}=\frac{9}{4}\ (4)$$

$$\frac{\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)^3}{\left(\sin^2 A+\cos^2\frac{A}{2}\right)^2}+\frac{\left(\sec^2\frac{B}{2}+\csc^2\frac{B}{2}\right)^3}{\left(\sin^2 B+\cos^2\frac{B}{2}\right)^2}+\frac{\left(\sec^2\frac{C}{2}+\csc^2\frac{C}{2}\right)^3}{\left(\sin^2 C+\cos^2\frac{C}{2}\right)^2}\overset{Radon}{\ge}$$

$$\ge\frac{\left(\sum\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)\right)^3}{\left(\sum\sin^2 A+\sum\cos^2\frac{A}{2}\right)^2}\overset{(2),(3),(4)}{\ge}\frac{(16)^3}{\left(\frac{9}{4}+\frac{9}{4}\right)^2}=\frac{256\times16\times4}{81}=\frac{16384}{81}$$

Equality holds iff ΔABC is equilateral.