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In ΔABC the following relationship holds:

$$\sum \frac{\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)^3}{\sin\frac{A}{2}+\sin\frac{B}{2}} \geq \frac{64\sqrt{3}}{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum \tan\frac{A}{2}+\sum \cot\frac{A}{2}=\frac{4R+r}{s}+\frac{s}{r} \overset{Doucet\ \&\ Mitrinovic}{\geq} \sqrt{3}+3\sqrt{3}=4\sqrt{3}\ (1)$$

$$\sum \sin\frac{A}{2} \overset{Jensen}{\leq} 3\sin\frac{\pi}{6}=\frac{3}{2} \quad (2)$$

$$f(x)=\sin\frac{x}{2}\ is\ concave \in (0,\pi)$$

$$\sum \frac{\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)^3}{\sin\frac{A}{2}+\sin\frac{B}{2}} \overset{Holder}{\geq} \frac{1}{3}\frac{\left(\sum\tan\frac{A}{2}+\sum\cot\frac{A}{2}\right)^3}{2\sum\sin\frac{A}{2}} \overset{(1)\&(2)}{\geq} \frac{\left(4\sqrt{3}\right)^3}{3.2.\frac{3}{2}}=\frac{192\sqrt{3}}{9}=\frac{64\sqrt{3}}{3}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$