ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationships holds:

$$1.\ \frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\geq 12\sqrt{3}r,$$

$$2.\ \frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\geq 72r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$1.\ \frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\geq 12\sqrt{3}r$$

$$\frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\overset{A-G}{\geq} 3\sqrt[3]{\frac{abc}{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}}\quad (1)$$

$$\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\leq\frac{1}{8};\ Let's\ prove\ it\ldots$$

$$a^2=(b-c)^2+4bc\sin^2\frac{A}{2}\rightarrow\begin{cases}a\geq 2\sin\frac{A}{2}\sqrt{bc}\\ b\geq \sin\frac{B}{2}\sqrt{ac}\\ c\geq 2\sin\frac{C}{2}\sqrt{ab}\end{cases}\rightarrow\ \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\leq\frac{1}{8}\quad (2)$$

From (1) and (2) we have:

$$\frac{a}{\sin\frac{A}{2}}+\frac{b}{\sin\frac{B}{2}}+\frac{c}{\sin\frac{C}{2}}\geq$$

$$\geq\left(3\sqrt[3]{8abc}=6\sqrt[3]{4R.S}\overset{R\geq 2r}{\geq}6\sqrt[3]{8r.S}\geq 12\sqrt[3]{r.3\sqrt{3}r^2}\right)=12r\sqrt{3}\ \ (True)$$

$$2.\ \frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\geq 72r^2$$

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\overset{A-G}{\overbrace{\geq}}3\sqrt[3]{\frac{(abc)^2}{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}}$$

If we use formulas ,we get :

$$\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\leq\frac{1}{8}\ \ and\ \ abc=4RS\ ,R\geq 2r\ \ Euler\ \ ,S\geq 3\sqrt{3}r^2\ \ Mitrinovici$$

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\overset{A-G}{\overbrace{\geq}}3\sqrt[3]{8(abc)^2}=$$

$$=\left(6\sqrt[3]{(4RS)^2}=6\sqrt[3]{\left(4.2r.3\sqrt{3}r^2\right)^2}\right)=72r^2\ (True)$$