In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{w_b^2(w_b^5+h_c^5)}+\frac{w_b^2}{h_c^2(h_c^5+m_a^5)}+\frac{h_c^2}{m_a^2(m_a^5+w_b^5)}\ge\frac{2^{10}r^6}{81R^6(81R^5-2560r^5)}$$

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$$\frac{1}{am_a}\sum_{cyc}a^2\ge 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\ge\frac{12}{\left(\sum_{cyc}a^2\right)^2}\Leftrightarrow$$

$$\left(\sum_{cyc}a^2\right)^2-3a^2(2b^2+2c^2-a^2)\ge 0\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2\left(2\sum_{cyc}a^2-3a^2\right)\ge 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-6a^2\sum_{cyc}a^2+9a^4\ge 0\Leftrightarrow\left(\sum_{cyc}a^2-3a^2\right)^2\ge 0$$

$$\Leftrightarrow(b^2+c^2-2a^2)^2\ge 0\to\text{true}\Rightarrow m_a^2\le\frac{\left(\sum_{cyc}a^2\right)^2}{12a^2}\text{ and analogs}\to(1)$$

$$\text{Again, }m_a^2\overset{?}{\le}\frac{b^3+c^3+abc}{4a}\Leftrightarrow a(2b^2+2c^2-a^2)\overset{?}{\le}b^3+c^3+abc$$

$$\Leftrightarrow\sum_{cyc}a^3+abc\overset{?}{\ge}2a(b^2+c^2)$$

$$\Leftrightarrow\sum_{cyc}(y+z)^3+\prod_{cyc}(y+z)\overset{?}{\ge}2(y+z)\left((z+x)^2+(x+y)^2\right)$$

$$\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow x+y+z=3s-2s=s\\ \Rightarrow a=y+z,b=z+x,c=x+y;x,y,z>0\end{matrix}\right)$$

$$\Leftrightarrow x^3+y^2z+yz^2\overset{?}{\ge}3xyz\to\text{true via A}-\text{G}\therefore m_a^2\le\frac{b^3+c^3+abc}{4a}$$

$$\Rightarrow m_a^3\overset{Panaitopol}{\le}\frac{b^3+c^3+abc}{4a}\cdot\frac{Rs}{a}=\frac{Rs}{4.16R^2r^2s^2}.b^2c^2(b^3+c^3+abc)$$

$$\Rightarrow m_a^3\le\frac{1}{64Rr^2s}\left(b^2c^2\left(\sum_{cyc}a^3+abc\right)-a^3b^2c^2\right)\Rightarrow m_a^5\le$$

$$\frac{1}{256Rr^2s}\cdot\left(b^2c^2\left(\sum_{cyc}a^3+abc\right)-a^3b^2c^2\right)\left(2\sum_{cyc}a^2-3a^2\right)\text{ and analogs}\Rightarrow$$

$$\sum_{cyc} m_a^5 \le \frac{1}{256Rr^2s}\cdot\left(\begin{array}{c}\left(2\sum_{cyc} a^2\right)\left(\left(\sum_{cyc} a^3 + abc\right)\left(\sum_{cyc} b^2c^2\right) - 16R^2r^2s^2(2s)\right)\\ -9a^2b^2c^2\left(\sum_{cyc} a^3 + abc\right) + 3a^2b^2c^2\sum_{cyc} a^3\end{array}\right)$$

$$\overset{\text{Goldstone}}{\le} \frac{1}{256Rr^2s}\left(\begin{array}{c}\left(2\sum_{cyc} a^2\right)\left(\begin{array}{c}\left(2s(s^2-6Rr-r^2)+4Rrs\right)(4R^2s^2)\\ -16R^2r^2s^2(2s)\end{array}\right)\\ -9(16R^2r^2s^2)\left(2s(s^2-4Rr-r^2)+4Rrs\right)\\ +3(16R^2r^2s^2).2s(s^2-6Rr-r^2)\end{array}\right) =$$

$$\frac{1}{256Rr^2s}\left(\begin{array}{c}\left(2\sum_{cyc} a^2\right)(8R^2s^3)(s^2-4Rr-7r^2) - 9(16R^2r^2s^2)\left(2s(s^2-4Rr-r^2)+4Rrs\right)\\ +3(16R^2r^2s^2).2s(s^2-6Rr-r^2)\end{array}\right)$$

$$\overset{\text{Leibnitz}}{\le} \frac{(18R^2)(8R^2s^3)(s^2-4Rr-7r^2) - 6(16R^2r^2s^2)\left(2s(s^2-4Rr-r^2)\right) - 9(64R^3r^3s^3)}{256Rr^2s}$$

$$= \frac{48R^2s^3\left((3R^2-4r^2)s^2 - r(12R^3+21R^2r-12Rr^2-12r^3)\right)}{256Rr^2s}$$

$$\overset{\text{Gerretsen}}{\le} \frac{48R^2s^3\left((3R^2-4r^2)(4R^2+4Rr+3r^2) - r(12R^3+21R^2r-12Rr^2-12r^3)\right)}{256Rr^2s}$$

$$\therefore \sum_{cyc} m_a^5 \le \frac{48R^2s^3}{256Rr^2s}.4R(3R^3-7Rr^2-r^3) \to (2)$$

$$\text{Now,} \frac{m_a^2}{w_b^2(w_b^5+h_c^5)} + \frac{w_b^2}{h_c^2(h_c^5+m_a^5)} + \frac{h_c^2}{m_a^2(m_a^5+w_b^5)}$$

$$\ge \frac{h_a^2}{m_b^2(m_b^5+m_c^5)} + \frac{h_b^2}{m_c^2(m_c^5+m_a^5)} + \frac{h_c^2}{m_a^2(m_a^5+m_b^5)} \overset{\text{via (1)}}{\ge}$$

$$\frac{48r^2s^2}{\left(\sum_{cyc} a^2\right)^2}\left(\frac{\frac{b^2}{a^2}}{m_b^5+m_c^5} + \frac{\frac{c^2}{b^2}}{m_c^5+m_a^5} + \frac{\frac{a^2}{c^2}}{m_a^5+m_b^5}\right) \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\ge} \frac{24r^2s^2\left(\sum_{cyc}\frac{b}{a}\right)^2}{81R^4.\sum_{cyc} m_a^5}$$

$$\overset{\substack{\text{via (2)}\\ \text{and}\\ \text{A-G}}}{\ge} \frac{9.24r^2s^2.256Rr^2s}{81R^4.48R^2s^3.4R(3R^3-7Rr^2-r^3)} \overset{?}{\ge} \frac{1024r^6}{81R^6(81R^5-2560r^5)}$$

$$\Leftrightarrow 729t^5 - 96t^3 + 224t - 23008 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(729t^4 + 1458t^3 + 2820t^2 + 5640t + 11504) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \frac{m_a^2}{w_b^2(w_b^5+h_c^5)} + \frac{w_b^2}{h_c^2(h_c^5+m_a^5)} + \frac{h_c^2}{m_a^2(m_a^5+w_b^5)} \ge \frac{2^{10}r^6}{81R^6(81R^5-2560r^5)}$$

$\forall \Delta ABC$," =" iff ΔABC is equilateral (QED)