ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, following relationship holds :

$$\sum_{cyc}\frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B}+\frac{c^8}{b^5\sin C}\right)\ge\frac{2}{R^2\sqrt{3}}$$

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$$\forall A',B',C'>0,(A'+B'),(B'+C'),(C'+A')\text{ form sides of a triangle}$$
$$(\because (A'+B')+(B'+C')>(C'+A')\text{ and analogs})$$
$$\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}\text{ form sides of a triangle with area F (say) and }16F^2$$
$$=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$
$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B}+\frac{c^8}{b^5\sin C}\right)+\frac{1}{a^5+b^5}\left(\frac{b^5}{c^2\sin C}+\frac{a^8}{c^5\sin A}\right)$$
$$+\frac{1}{b^5+c^5}\left(\frac{c^5}{a^2\sin A}+\frac{b^8}{a^5\sin B}\right)$$
$$=2R.\frac{c^5a^5}{a^5b^5+b^5c^5}.\frac{b^5}{c^5a^5}.\left(\frac{a^5}{b^3}+\frac{c^7}{b^5}\right)+2R.\frac{a^5b^5}{c^5a^5+b^5c^5}.\frac{c^5}{a^5b^5}.\left(\frac{b^5}{c^3}+\frac{a^7}{c^5}\right)$$
$$+2R.\frac{b^5c^5}{c^5a^5+a^5b^5}.\frac{a^5}{b^5c^5}.\left(\frac{c^5}{a^3}+\frac{b^7}{a^5}\right)$$
$$=2R.\frac{b^5c^5}{c^5a^5+a^5b^5}.\left(\frac{a^2}{b^5}+\frac{b^2}{c^5}\right)+2R.\frac{c^5a^5}{a^5b^5+b^5c^5}.\left(\frac{b^2}{c^5}+\frac{c^2}{a^5}\right)$$
$$+2R.\frac{a^5b^5}{c^5a^5+b^5c^5}.\left(\frac{c^2}{a^5}+\frac{a^2}{b^5}\right)$$
$$=2R.\frac{x}{y+z}(B'+C')+2R.\frac{y}{z+x}(C'+A')+2R.\frac{z}{x+y}(A'+B')$$

$$\left(x=b^5c^5, y=c^5a^5, z=a^5b^5, A'=\frac{c^2}{a^5}, B'=\frac{a^2}{b^5}, C'=\frac{b^2}{c^5}\right)$$

$$=2R.\frac{x}{y+z}.\sqrt{B'+C'}^2+2R.\frac{y}{z+x}.\sqrt{C'+A'}^2+2R.\frac{z}{x+y}.\sqrt{A'+B'}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2R.2\sqrt{\sum_{cyc}A'B'.\frac{\sqrt{3}}{2}}$$

$$=2R.\sqrt{3\sum_{cyc}\left(\frac{c^2}{a^5}.\frac{a^2}{b^5}\right)}\overset{A-G}{\geq}6R.\sqrt[6]{\frac{c^2}{a^5}.\frac{a^2}{b^5}.\frac{a^2}{b^5}.\frac{b^2}{c^5}.\frac{b^2}{c^5}.\frac{c^2}{a^5}}=6R.\sqrt[6]{\frac{1}{a^6b^6c^6}}=\frac{6R}{4Rrs}$$

$$=\frac{3}{2rs}\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\geq}\frac{6}{R.3\sqrt{3}R}\therefore\frac{1}{a^5+c^5}\left(\frac{a^5}{b^2\sin B}+\frac{c^8}{b^5\sin C}\right)$$

$$+\frac{1}{a^5+b^5}\left(\frac{b^5}{c^2\sin C}+\frac{a^8}{c^5\sin A}\right)+\frac{1}{b^5+c^5}\left(\frac{c^5}{a^2\sin A}+\frac{b^8}{a^5\sin B}\right)\geq\frac{2}{R^2.\sqrt{3}}$$

$\forall\,\Delta\,ABC$," = " iff $\Delta\,ABC$ is equilateral (QED)