ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{n_a}{m_b}+\frac{p_b}{w_c}+\frac{g_c}{h_a}\geq\frac{6r}{R}$$

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$$\frac{n_a}{m_b}+\frac{p_b}{w_c}+\frac{g_c}{h_a}\geq\frac{h_a}{m_b}+\frac{h_b}{m_c}+\frac{h_c}{m_a}\overset{A-G}{\geq}3.\sqrt[3]{\frac{h_ah_bh_c}{m_am_bm_c}}\overset{m_am_bm_c\leq\frac{Rs^2}{2}}{\geq}3.\sqrt[3]{\frac{2r^2s^2}{R.\frac{Rs^2}{2}}}$$

$$=3.\sqrt[3]{\frac{4r^2}{R^2}}\overset{?}{\geq}\frac{6r}{R}\Leftrightarrow\frac{4r^2}{R^2}\overset{?}{\geq}\frac{8r^3}{R^3}\Leftrightarrow R\overset{?}{\geq}2r\to\text{true via Euler}\therefore\frac{n_a}{m_b}+\frac{p_b}{w_c}+\frac{g_c}{h_a}\geq\frac{6r}{R}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Proof of $m_am_bm_c\leq\frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\text{Now, }\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\to(2)$$

$$\text{Also, }\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)=$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\to(3)\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(\begin{array}{c} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{array}\right)$$

$$= \frac{1}{16}(s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3)$$

$$\leq \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underset{(*)}{\underbrace{\overset{\text{Gerretsen}}{\geq}}} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underset{(**)}{\underbrace{\overset{\text{Gerretsen}}{\leq}}} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(*), (**) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$$

$$\geq 20rs^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R+r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underset{(***)}{\underbrace{\overset{\text{Gerretsen}}{\geq}}} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underset{(****)}{\underbrace{\overset{\text{Gerretsen}}{\leq}}} 27r^2(4R^2 + 4Rr + 3r^2)$$

$(***), (****) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t + 309) + 648\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \leq \frac{Rs^2}{2} \text{ (QED)}$$