ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{(h_a^3+2h_aw_b(h_a+w_b)+w_b^3)^3}{h_a^4+2h_aw_b(h_a^2+w_b^2)+w_b^4}+\frac{(w_b^3+2w_bm_c(w_b+m_c)+m_c^3)^3}{w_b^4+2w_bm_c(w_b^2+m_c^2)+m_c^4}+\frac{(m_c^3+2m_ch_a(m_c+h_a)+h_a^3)^3}{m_c^4+2m_ch_a(m_c^2+h_a^2)+h_a^4}\geq\frac{9.6^6.r^9}{R(9R^3-64r^3)}$$

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$$\sum_{cyc} a^4 \overset{?}{\leq} 54R^3(R-r)$$

$$\Leftrightarrow 2\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-16r^2s^2 \overset{?}{\leq} 54R^3(R-r)$$

$$\Leftrightarrow s^4-(8Rr+6r^2)s^2 \underset{(1)}{\overset{?}{\leq}} 27R^4-27R^3r-16R^2r^2-8Rr^3-r^4$$

$$\text{Now, LHS of (1)} \overset{\text{Gerretsen}}{\leq} (4R^2-4Rr-3r^2)s^2 \overset{\text{Gerretsen}}{\leq}$$

$$(4R^2-4Rr-3r^2)(4R^2+4Rr+3r^2) \overset{?}{\leq} 27R^4-27R^3r-16R^2r^2-8Rr^3-r^4$$

$$\Leftrightarrow 11t^4-27t^3+16t+8 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(11t^2+17t+24)+44\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow \text{(1) is true} \therefore \sum_{cyc} a^4 \overset{(*)}{\leq} 54R^3(R-r)$$

$$\frac{(h_a^3+2h_aw_b(h_a+w_b)+w_b^3)^3}{h_a^4+2h_aw_b(h_a^2+w_b^2)+w_b^4}+\frac{(w_b^3+2w_bm_c(w_b+m_c)+m_c^3)^3}{w_b^4+2w_bm_c(w_b^2+m_c^2)+m_c^4}+\frac{(m_c^3+2m_ch_a(m_c+h_a)+h_a^3)^3}{m_c^4+2m_ch_a(m_c^2+h_a^2)+h_a^4}\geq\sum_{cyc}\frac{(h_a^3+2h_ah_b(h_a+h_b)+h_b^3)^3}{m_a^4+2m_am_b(m_a^2+m_b^2)+m_b^4}$$

$$\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}(3h_ah_b(h_a+h_b))\right)^3}{3\sum_{cyc}\left(m_a^4+(m_a^2+m_b^2)^2+m_b^4\right)}=\frac{\left(3.\frac{2r^2s^2}{R}.\sum_{cyc}\left(\frac{h_a+h_b}{h_c}\right)\right)^3}{3(4\sum_{cyc}m_a^4+2\sum_{cyc}m_b^2m_c^2)}$$

$$\overset{\substack{\text{Gerretsen + Euler}\\ and\\ \text{A-G}}}{\geq}\frac{\left(3.\frac{r^2.27Rr}{R}.6\right)^3}{3.6\sum_{cyc}m_a^4}=\frac{3^3.3^9.2^3.3^3.r^9}{\frac{3^3.6}{16}.\sum_{cyc}a^4}\overset{\text{via }(*)}{\geq}\frac{3^{11}.2^6.r^9}{2.3^3.R^3(R-r)}$$

$$\overset{?}{\geq}\frac{9.6^6.r^9}{R(9R^3-64r^3)}=\frac{3^8.2^6.r^9}{R(9R^3-64r^3)}\Leftrightarrow 9R^3-64r^3\overset{?}{\geq}2R^2(R-r)$$

$$\Leftrightarrow 7t^3+2t^2-64\overset{?}{\geq}0\Leftrightarrow(t-2)(7t^2+16t+32)\overset{?}{\geq}0\to\text{true}$$

$$\therefore\frac{(h_a^3+2h_aw_b(h_a+w_b)+w_b^3)^3}{h_a^4+2h_aw_b(h_a^2+w_b^2)+w_b^4}+\frac{(w_b^3+2w_bm_c(w_b+m_c)+m_c^3)^3}{w_b^4+2w_bm_c(w_b^2+m_c^2)+m_c^4}+\frac{(m_c^3+2m_ch_a(m_c+h_a)+h_a^3)^3}{m_c^4+2m_ch_a(m_c^2+h_a^2)+h_a^4}\geq\frac{9.6^6.r^9}{R(9R^3-64r^3)}$$

$\forall\,\Delta$ ABC,″ =″ iff Δ ABC is equilatera$l$ (QED)