ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{\left(h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+\frac{\left(w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$
$$+\frac{\left(m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5}\geq\frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$
$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)=2\sum_{cyc}r_a^5+2\sum_{cyc}\left(r_b^2r_c^2\left(\sum_{cyc}r_a-r_a\right)\right)$$
$$=2\left(\left(\sum_{cyc}r_a\right)^5-5\left(\sum_{cyc}r_a^2+\sum_{cyc}r_ar_b\right)\prod_{cyc}(r_b+r_c)\right)$$
$$+2(4R+r)\left(\left(\sum_{cyc}r_ar_b\right)^2-2rs^2\left(\sum_{cyc}r_a\right)\right)-2rs^2\left(\sum_{cyc}r_ar_b\right)$$
$$\overset{\text{via (i) and analogs}}{=}2\left((4R+r)^5-5\left((4R+r)^2-2s^2+s^2\right).64R^2.\frac{s^2}{16R^2}\right)$$
$$+2(4R+r)\left(s^4-2rs^2(4R+r)\right)-2rs^4$$
$$=2\left((4R+r)^5+24Rs^4-\left(320R^3+192R^2r+36Rr^2+2r^3\right)\right)s^2$$
$$\overset{\text{Gerretsen}}{\leq}2\begin{pmatrix}(4R+r)^5+24R\left(4R^2+4Rr+3r^2\right)s^2\\-\left(320R^3+192R^2r+36Rr^2+2r^3\right)s^2\end{pmatrix}$$
$$=2\left((4R+r)^5-\left(224R^3+96R^2r-36Rr^2+2r^3\right)s^2\right)\overset{\text{Gerretsen}}{\leq}$$
$$2(4R+r)^5-2\left(224R^3+96R^2r-36Rr^2+2r^3\right)\left(16R-5r^2\right)$$
$$\therefore\boxed{\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)\leq2\left(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5\right)}$$
$$\to(1)$$

$$\text{Again,}\sum_{cyc}\left(h_b^4+2h_bh_c\left(h_b^2+h_c^2\right)+h_c^4\right)\overset{A-G}{\geq}$$

$$2\sum_{cyc} h_b h_c\left(h_b^2+h_c^2+h_b h_c\right) \overset{A-G}{\geq} \frac{6h_a^2h_b^2h_c^2}{4r^2s^2}\cdot\sum_{cyc} a^2 \geq \frac{6.4r^4s^4}{4R^2r^2s^2}\cdot\frac{4s^2}{3}$$

$$= \frac{8r^2s^4}{R^2} \overset{\text{Gerretsen + Euler}}{\geq} \frac{2r^2}{R^2}(27Rr)^2$$

$$\boxed{\sum_{cyc}\left(h_b^4+2h_b h_c\left(h_b^2+h_c^2\right)+h_c^4\right) \geq 2.729r^4} \to (2)$$

$$\therefore \frac{\left(h_a^4+2h_a w_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+\frac{\left(w_b^4+2w_b m_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$

$$+\frac{\left(m_c^4+2m_c h_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5} \geq \sum_{cyc}\frac{\left(h_b^4+2h_b h_c\left(h_b^2+h_c^2\right)+h_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$

$$\overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc}\left(h_b^4+2h_b h_c\left(h_b^2+h_c^2\right)+h_c^4\right)\right)^n}{3^{n-2}.\left(\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)\right)} \overset{via\ (1)\ and\ (2)}{\geq}$$

$$\frac{\left(2.729r^4\right)^n}{3^{n-2}.2.\left(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5\right)} \overset{?}{\geq} \frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}$$

$$\Leftrightarrow (3^7)(9R^3-64r^3)(3R^2-8r^2) \overset{?}{\geq}$$
$$32\left(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5\right)$$

$$\Leftrightarrow 26281t^5+73728t^4-164632t^3-458816t^2+6144t+1119392 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(26281t^3+178852t^2+445652t+608384)+657072\right) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{\left(h_a^4+2h_a w_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+$$

$$\frac{\left(w_b^4+2w_b m_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}+\frac{\left(m_c^4+2m_c h_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5}$$

$$\geq \frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$