ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationships hold :

$$1.\ \frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{4.3^{n-1}.r^n}{R^2}\ and$$

$$2.\ \frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{(3r)^n}{R}$$

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$$\frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{m_a h_a^n}{m_a m_b m_c}+\frac{m_b h_b^n}{m_b m_c m_a}+\frac{m_c h_c^n}{m_c m_a m_b}\overset{m_a m_b m_c\le\frac{Rs^2}{2}}{\ge}$$

$$\frac{2}{Rs^2}\cdot\sum_{cyc}m_a h_a^n\overset{Chebyshev}{\ge}\frac{2}{3Rs^2}\cdot\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}h_a^n\right)$$

$$(\because \text{WLOG assuming } a\ge b\ge c\Rightarrow m_a\le m_b\le m_c \text{ and } h_a^n\le h_b^n\le h_c^n)$$

$$\overset{\substack{Tereshin\\+\\Holder}}{\ge}\frac{2}{3Rs^2}\cdot\sum_{cyc}\frac{b^2+c^2}{4R}\cdot\frac{1}{3^{n-1}}\cdot\left(\sum_{cyc}h_a\right)^n=\frac{2}{3Rs^2}\cdot\frac{\sum_{cyc}a^2}{2R}\cdot\frac{1}{3^{n-1}}\cdot\left(2rs\sum_{cyc}\frac{1}{a}\right)^n$$

$$\overset{Bergstrom}{\ge}\frac{\sum_{cyc}a^2}{R^2s^2}\cdot\frac{1}{3^n}\cdot\left(2rs.\frac{9}{2s}\right)^n=\frac{\sum_{cyc}a^2}{R^2s^2}\cdot\frac{1}{3^n}.3^{2n}.r^n\overset{?}{\ge}\frac{4.3^{n-1}.r^n}{R^2}\Leftrightarrow 3\sum_{cyc}a^2\overset{?}{\ge}4s^2$$

$$\Leftrightarrow 3\sum_{cyc}a^2\overset{?}{\ge}\left(\sum_{cyc}a\right)^2\to \text{true}$$

$$\therefore\frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{4.3^{n-1}.r^n}{R^2}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral}$$

$$\text{Again},\frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{h_a^n}{m_b+m_c}+\frac{w_b^n}{m_c+m_a}+\frac{m_c^n}{m_a+m_b}$$

$$\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum_{cyc}h_a^n\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)$$

$$\left(\because \text{WLOG assuming } a\ge b\ge c\Rightarrow\frac{1}{m_b+m_c}\le\frac{1}{m_c+m_a}\le\frac{1}{m_a+m_b}\text{ and } h_a^n\le h_b^n\le h_c^n\right)$$

$$\overset{\substack{Bergstrom\\+\\Holder}}{\ge}\frac{1}{3.3^{n-1}}\cdot\left(\sum_{cyc}h_a\right)^n\cdot\frac{9}{2\sum_{cyc}m_a}\overset{Leuenbrger}{\ge}\frac{1}{3^n}\cdot\left(2rs\sum_{cyc}\frac{1}{a}\right)^n\cdot\frac{9}{2(4R+r)}\overset{\substack{Bergstrom\\+\\Euler}}{\ge}$$

$$\frac{1}{3^n}\cdot\left(2rs\frac{9}{2s}\right)^n\cdot\frac{9}{2\left(\frac{9R}{2}\right)}=\frac{(3r)^n}{R}\therefore\frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{(3r)^n}{R}$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

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Proof of $m_a m_b m_c \le \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\ -576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

Now, LHS of $(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$

$-r^3(4R+r)^3\overset{?}{\le}0$

$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underset{(\bullet\bullet)}{\underbrace{\overset{?}{\ge}}}20rs^4$

Now, LHS of $(\bullet\bullet)\underset{(a)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}}s^2(16Rr-5r^2)(8R-16r)$

$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$ and

RHS of $(\bullet\bullet)\underset{(b)}{\underbrace{\overset{\text{Geretsen}}{\le}}}20rs^2(4R^2+4Rr+3r^2)$

$(a),(b)\Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$

$\ge 20rs^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$

$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\ge}27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet)\underset{(c)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underset{(d)}{\underbrace{\overset{\text{Geretsen}}{\le}}}27r^2(4R^2+4Rr+3r^2)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\ge 27r^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow 224t^3-587t^2+308t-60\ge 0\left(\text{where } t=\frac{R}{r}\right)$

$\Leftrightarrow (t-2)((t-2)(224t+309)+648)\ge 0\to$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$

$\Rightarrow(\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\le\frac{Rs^2}{2}$ (QED)