ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$ and $\forall\, n \in \mathbb{N} : n \geq 2$, the following relationship hold :

$$\frac{h_a^n}{w_b^2 m_c(w_b^2+m_c^2)}+\frac{w_b^n}{m_c^2 h_a(m_c^2+h_a^2)}+\frac{m_c^n}{h_a^2 w_b(h_a^2+w_b^2)} \geq \frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

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$$\frac{h_a^n}{w_b^2 m_c(w_b^2+m_c^2)}+\frac{w_b^n}{m_c^2 h_a(m_c^2+h_a^2)}+\frac{m_c^n}{h_a^2 w_b(h_a^2+w_b^2)}$$

$$\geq \frac{h_a^n}{m_b^2 m_c(m_b^2+m_c^2)}+\frac{h_b^n}{m_c^2 m_a(m_c^2+m_a^2)}+\frac{h_c^n}{m_a^2 m_b(m_a^2+m_b^2)}$$

$$\overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^n}{3^{n-2}.\sum_{cyc}\left(m_b^2 m_c\left(\sum_{cyc} m_a^2 - m_a^2\right)\right)}$$

$$= \frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.\left(\left(\sum_{cyc} m_a^2\right)\left(\sum_{cyc} m_b^2 m_c\right) - m_a m_b m_c \sum_{cyc} m_a m_b\right)}$$

$$\overset{\text{Bergstrom + A-G}}{\geq} \frac{\left(2rs.\frac{9}{2s}\right)^n}{3^{n-2}.\left(\frac{3}{4}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} m_a^3\right) - m_a m_b m_c.\sum_{cyc} h_a h_b\right)}$$

$$\overset{\text{Leibnitz}}{\geq} \frac{3^{2n}.r^n}{3^{n-2}.\left(\frac{3}{4}.9R^2.\left(\left(\sum_{cyc} m_a\right)^3 - 3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right) - m_a m_b m_c.\sum_{cyc}\frac{bc.ca}{4R^2}\right)}$$

$$\overset{\text{Leuenberger + Cesaro}}{\geq} \frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3 - 24 m_a m_b m_c\right) - m_a m_b m_c.\frac{4Rrs(2s)}{4R^2}}$$

$$\geq \frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3 - 24 h_a h_b h_c\right) - h_a h_b h_c.\frac{2rs^2}{R}}$$

$$= \frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3 - 24.\frac{2r^2s^2}{R}\right) - \frac{2r^2s^2}{R}.\frac{2rs^2}{R}}$$

$$\overset{\text{Gerretsen + Euler}}{\geq} \frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3 - 24.\frac{r^2.27Rr}{R}\right) - \frac{r^2.27Rr}{R}.\frac{r.27Rr}{R}}$$

$$= \frac{4.3^{n-1}.r^n}{R^2((4R+r)^3 - 24.27r^3) - 108r^5} \overset{?}{\geq} \frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

$$\Leftrightarrow 27(81R^5 - 2560r^5) \overset{?}{\geq} 4\left(R^2((4R+r)^3 - 24.27r^3) - 108r^5\right)$$

$$\Leftrightarrow 1931t^5 - 192t^4 - 48t^3 + 2588t^2 - 68688 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(1931t^4+3670t^3+7292t^2+17172t+34344\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \frac{h_a^n}{w_b^2 m_c\left(w_b^2+m_c^2\right)}+\frac{w_b^n}{m_c^2 h_a\left(m_c^2+h_a^2\right)}+\frac{m_c^n}{h_a^2 w_b\left(h_a^2+w_b^2\right)} \geq \frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

$\forall\ \Delta\ ABC$ *and* $\forall\ n \in \mathbb{N} : n \geq 2,'' ='' $ iff $\Delta\ ABC$ is equi*la*ter*al* (QED)