ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a}{w_b}+\frac{w_b}{h_c}+\frac{h_c}{m_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a}{w_b}+\frac{w_b}{h_c}+\frac{h_c}{m_a}\le\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\overset{\text{Panaitopol}}{\le}\frac{R}{2r}\left(\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\right)$$

$$=\frac{R}{2r}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{b}\right)\overset{\text{CBS}}{\le}\frac{R}{2r}*\sqrt{\sum_{cyc}a^2}*\sqrt{\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Goldstone}}}{\le}\left(\frac{R}{2r}\right)(3R)*\sqrt{\frac{4R^2s^2}{16R^2r^2s^2}}$$

$$=\left(\frac{R}{2r}\right)\left(\frac{3R}{2r}\right)\overset{?}{\le}\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)=\frac{3(9R^3-64r^3)}{8r^3}\Leftrightarrow 9R^3-2R^2r-64r^3\overset{?}{\ge}0$$

$$\Leftrightarrow 8\left(R^3-(2r)^3\right)+R^2(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\frac{m_a}{w_b}+\frac{w_b}{h_c}+\frac{h_c}{m_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$