ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ n $\in \mathbb{N}$: n $\geq$ 2, the following relationship hold :

$$\frac{h_a^n}{r_b^2 r_c(r_b^3+r_c^3)}+\frac{w_b^n}{r_c^2 r_a(r_c^3+r_a^3)}+\frac{m_c^n}{r_a^2 r_b(r_a^3+r_b^3)} \geq \frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}$$

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$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\sum_{cyc} r_b^2 r_c(r_b^3+r_c^3)=\sum_{cyc}\left(r_b^2 r_c\left(\sum_{cyc} r_a^3-r_a^3\right)\right)$$

$$=\left(\sum_{cyc} r_a^3\right)\left(\sum_{cyc} r_b^2 r_c\right)-r_a r_b r_c\left(\sum_{cyc} r_a^2 r_b\right)\overset{A-G}{\leq}\left(\sum_{cyc} r_a^3\right)^2-3r_a^2 r_b^2 r_c^2$$

$$=\left((4R+r)^3-3\prod_{cyc}(r_b+r_c)\right)^2-3r^2s^4$$

$$\overset{\text{via (i) and analogs}}{=}\left((4R+r)^3-3.64R^3.\frac{s^2}{16R^2}\right)^2-3r^2s^4$$

$$=(4R+r)^6+(144R^2-3r^2)s^4-24R(4R+r)^3s^2 \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq}$$

$$(4R+r)^4.\frac{81R^2}{4}+\left((144R^2-3r^2).\frac{27R^2}{4}-24R(4R+r)^3\right)s^2$$

$$=(4R+r)^4.\frac{81R^2}{4}-\frac{3s^2}{4}(752R^4+1536R^3r+411R^2r^2+32Rr^3)\overset{\text{Gerretsen + Euler}}{\leq}$$

$$(4R+r)^4.\frac{81R^2}{4}-\frac{3.27Rr}{8}(752R^4+1536R^3r+411R^2r^2+32Rr^3)$$

$$\therefore \boxed{32\sum_{cyc} r_b^2 r_c(r_b^3+r_c^3)\overset{(*)}{\leq} 81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)}$$

$$\text{Now, }\frac{h_a^n}{r_b^2 r_c(r_b^3+r_c^3)}+\frac{w_b^n}{r_c^2 r_a(r_c^3+r_a^3)}+\frac{m_c^n}{r_a^2 r_b(r_a^3+r_b^3)} \geq$$

$$\frac{h_a^n}{r_b^2 r_c(r_b^3+r_c^3)}+\frac{h_b^n}{r_c^2 r_a(r_c^3+r_a^3)}+\frac{h_c^n}{r_a^2 r_b(r_a^3+r_b^3)} \overset{\text{Holder}}{\geq} \frac{32\left(\sum_{cyc} h_a\right)^n}{3^{n-2}.32\sum_{cyc} r_b^2 r_c(r_b^3+r_c^3)}$$

$$\overset{via\ (*)}{\geq} \frac{32\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)} \overset{Bergstrom}{\geq}$$

$$\frac{32\left(2rs.\frac{9}{2s}\right)^n}{3^{n-2}.81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)} \overset{?}{\geq}$$

$$\frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}$$

$$\Leftrightarrow \frac{2187}{81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)}$$

$$\overset{?}{\geq} \frac{1}{3(9R^3-64r^3)^2-128r^6} \Leftrightarrow 81\left(9R^3-64r^3\right)^2-27.128r^6 \overset{?}{\geq}$$

$$8R^2(4R+r)^4-4Rr\left(752R^4+1536R^3r+411R^2r^2+32Rr^3\right)$$

$$\Leftrightarrow 4513t^6+960t^5+5376t^4-91796t^3+120t^2+328320 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(4513t^4+19012t^3+63372t^2+85644t+89208)+14256\right)$$

$$\overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \therefore \frac{h_a^n}{r_b^2r_c(r_b^3+r_c^3)}+\frac{w_b^n}{r_c^2r_a(r_c^3+r_a^3)}+\frac{m_c^n}{r_a^2r_b(r_a^3+r_b^3)}$$

$$\geq \frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$