ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta\ ABC$ and $\forall\ n \in \mathbb{N} : n \geq 2$, the fol$l$owing rel$a$tionship hol$d$:

$$\frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b} \geq \frac{16.3^{n-3}.r^n}{3(3R^2 - 8r^2)^2 + 3R^4 - 64r^4}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b} \geq$$

$$\frac{h_a^n}{r_b^3 r_c + m_b^3 m_c} + \frac{h_b^n}{r_c^3 r_a + m_c^3 m_a} + \frac{h_c^n}{r_a^3 r_b + m_a^3 m_b} \overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^n}{3^{n-2}.\left(\sum_{cyc} r_a^3 r_b + \sum_{cyc} m_a^3 m_b\right)}$$

$$\overset{\text{Vasc}}{\geq} \frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.\left(\frac{\left(\sum_{cyc} r_a^2\right)^2 + \left(\sum_{cyc} m_a^2\right)^2}{3}\right)} \overset{\text{Bergstrom}}{\geq} \frac{\left(2rs.\frac{9}{2s}\right)^n}{3^{n-3}.\left(((4R+r)^2 - 2s^2)^2 + \frac{9}{16}\left(\sum_{cyc} a^2\right)^2\right)}$$

$$\overset{\substack{\text{Gerretsen}\\ and\\ \text{Leibnitz}}}{\geq} \frac{3^{n+3}.r^n}{\left((4R+r)^2 - 2(16Rr - 5r^2)\right)^2 + \frac{9}{16}.81R^4}$$

$$= \frac{16.3^{n+3}.r^n}{16(16R^2 - 24Rr + 11r^2)^2 + 729R^4} \overset{?}{\geq} \frac{16.3^{n-3}.r^n}{3(3R^2 - 8r^2)^2 + 3R^4 - 64r^4}$$

$$\Leftrightarrow 729\left(3(3R^2 - 8r^2)^2 + 3R^4 - 64r^4\right) \overset{?}{\geq} 16\left(16R^2 - 24Rr + 11r^2\right)^2 + 729R^4$$

$$\Leftrightarrow 17045t^4 + 12288t^3 - 119824t^2 + 8448t + 91376 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(17045t^2 + 80468t + 133868) + 222048\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b}$$

$$\geq \frac{16.3^{n-3}.r^n}{3(3R^2 - 8r^2)^2 + 3R^4 - 64r^4}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equil}a\text{teral (QED)}$$