ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall\, n \in \mathbb{N} : n \geq 2$, the following relationship holds :

$$\frac{h_a^n}{r_a^8 + 2r_a^3r_b^3(r_a^2+r_b^2)+r_b^8} + \frac{w_b^n}{r_b^8 + 2r_b^3r_c^3(r_b^2+r_c^2)+r_c^8} + \frac{m_c^n}{r_c^8 + 2r_c^3r_a^3(r_c^2+r_a^2)+r_a^8} \geq \frac{128.3^{n-8}.r^n}{(81R^5-2560r^5)(9R^3-64r^3)}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0, \sum_{cyc} x^8 + \sum_{cyc} x^3y^3(x^2+y^2) = \left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x^5\right) \to (1)$$

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now,} \sum_{cyc} r_a^3 = \left(\sum_{cyc} r_a\right)^3 - 3(r_a+r_b)(r_b+r_c)(r_c+r_a) \overset{\text{via (i) and analogs}}{=}$$

$$(4R+r)^3 - 3.64R^3.\frac{s^2}{16R^2} \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\frac{9R}{2}\right)^3 - 24r.27r^2 \therefore \sum_{cyc} r_a^3 \overset{(2)}{\leq} \frac{81}{8}(9R^3-64r^3)$$

$$\text{Again,} \sum_{cyc} r_a^5 \overset{\text{via (2)}}{\leq} \left(\sum_{cyc} r_a^2\right)\left(\sum_{cyc} r_a^3\right) - \sum_{cyc} r_a^2r_b^2(r_a+r_b)$$

$$= \left((4R+r)^2 - 2s^2\right).\frac{81}{8}(9R^3-64r^3) - (4R+r)\left(\sum_{cyc} r_a^2r_b^2\right) + r_ar_br_c\sum_{cyc} r_ar_b$$

$$\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\left(\frac{9R}{2}\right)(4R+r) - 2.27r^2\right).\frac{81}{8}(9R^3-64r^3) - \frac{(4R+r)}{3}.\left(\sum_{cyc} r_ar_b\right)^2$$

$$+rs^4 \overset{\text{Mitrinovic}}{\leq} \left(\left(\frac{9R}{2}\right)(4R+r) - 2.27r^2\right).\frac{81}{8}(9R^3-64r^3) - \frac{(4R+r)}{3}.729r^4$$

$$+r.\frac{729R^4}{16} \overset{\text{Euler}}{\leq} 729\left(\left(\frac{R(4R+r)}{2} - 6r^2\right)\left(\frac{9R^3-64r^3}{8}\right) - 3r^5 + \frac{R^4r}{16}\right)$$

$$= \frac{729\left(\left(R(4R+r)-12r^2\right)(9R^3-64r^3) - 48r^5 + R^4r\right)}{16} \overset{?}{\leq} \frac{729(81R^5-2560r^5)}{32}$$

$$\Leftrightarrow 9t^5 - 20t^4 + 216t^3 + 512t^2 + 128t - 4000 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

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$$\Leftrightarrow (t-2)\left(8t^4+t^3(t-2)+212t^2+936t+2000\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \sum_{cyc} r_a^5 \overset{(3)}{\leq} \frac{729\left(81R^5-2560r^5\right)}{32}$$

$$\text{Also,} \frac{h_a^n}{r_a^8+2r_a^3r_b^3\left(r_a^2+r_b^2\right)+r_b^8}+\frac{w_b^n}{r_b^8+2r_b^3r_c^3\left(r_b^2+r_c^2\right)+r_c^8}$$

$$+\frac{m_c^n}{r_c^8+2r_c^3r_a^3\left(r_c^2+r_a^2\right)+r_a^8} \geq \frac{h_a^n}{r_a^8+2r_a^3r_b^3\left(r_a^2+r_b^2\right)+r_b^8}+\frac{h_b^n}{r_b^8+2r_b^3r_c^3\left(r_b^2+r_c^2\right)+r_c^8}$$

$$+\frac{h_c^n}{r_c^8+2r_c^3r_a^3\left(r_c^2+r_a^2\right)+r_a^8} \overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^n}{3^{n-2}.\sum_{cyc}\left(r_b^8+2r_b^3r_c^3\left(r_b^2+r_c^2\right)+r_c^8\right)}$$

$$= \frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{2.3^{n-2}.\left(\sum_{cyc} r_a^8+\sum_{cyc} r_a^3r_b^3\left(r_a^2+r_b^2\right)\right)} \overset{\substack{\text{Bergstrom} \\ \text{and} \\ \text{via (1)}}}{\geq} \frac{\left(2rs.\frac{9}{2s}\right)^n}{2.3^{n-2}.\left(\sum_{cyc} r_a^3\right)\left(\sum_{cyc} r_a^5\right)} \overset{\text{via (2) and (3)}}{\geq}$$

$$\frac{3^{n+2}.r^n}{2.\frac{81}{8}\left(9R^3-64r^3\right).\frac{729\left(81R^5-2560r^5\right)}{32}} = \frac{128.3^{n-8}.r^n}{\left(81R^5-2560r^5\right)\left(9R^3-64r^3\right)}$$

$\forall \Delta ABC$," = " iff ΔABC is equilateral (QED)