ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

1. $$\frac{h_a w_b}{h_a + w_b + 2m_c} + \frac{w_b m_c}{w_b + m_c + 2h_a} + \frac{m_c h_a}{m_c + h_a + 2w_b} \le \frac{9R}{8}$$

2. $$\frac{27r^3}{R} \le \frac{h_a {w_b}^2}{h_a + w_b} + \frac{w_b {m_c}^2}{w_b + m_c} + \frac{m_c {h_a}^2}{m_c + h_a} \le \frac{27R^2}{8}$$

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1. By CBS inequality, we have

$$\frac{h_a w_b}{h_a + w_b + 2m_c} + \frac{w_b m_c}{w_b + m_c + 2h_a} + \frac{m_c h_a}{m_c + h_a + 2w_b}$$
$$\le \frac{1}{4}\left(\frac{h_a w_b}{h_a + m_c} + \frac{h_a w_b}{w_b + m_c}\right) + \frac{1}{4}\left(\frac{w_b m_c}{w_b + h_a} + \frac{w_b m_c}{m_c + h_a}\right) + \frac{1}{4}\left(\frac{m_c h_a}{m_c + w_b} + \frac{m_c h_a}{h_a + w_b}\right)$$
$$= \frac{h_a + w_b + m_c}{4} \overset{h_a \le m_a \,\&\, w_b \le m_b}{\le} \frac{m_a + m_b + m_c}{4} \overset{Gotman}{\le} \frac{1}{4}\cdot\frac{9R}{2} = \frac{9R}{8},$$

as desired. Equality holds iff ΔABC is equilateral.

2. By AM − GM inequality, we have

$$\frac{h_a {w_b}^2}{h_a + w_b} + \frac{w_b {m_c}^2}{w_b + m_c} + \frac{m_c {h_a}^2}{m_c + h_a} \le \frac{w_b(h_a + w_b)}{4} + \frac{m_c(w_b + m_c)}{4} + \frac{h_a(m_c + h_a)}{4}$$
$$= \frac{(h_a w_b + w_b m_c + m_c h_a) + \left({h_a}^2 + {w_b}^2 + {m_c}^2\right)}{4} \le \frac{{h_a}^2 + {w_b}^2 + {m_c}^2}{2}$$
$$\overset{h_a \le m_a \,\&\, w_b \le m_b}{\le} \frac{{m_a}^2 + {m_b}^2 + {m_c}^2}{2} = \frac{3\left(a^2 + b^2 + c^2\right)}{8} \overset{Leibniz}{\le} \frac{3.9R^2}{8} = \frac{27R^2}{8}.$$

Now, by using AM − GM inequality and $h_a \le w_a \le m_a$ (and analogs), we have

$$\frac{h_a {w_b}^2}{h_a + w_b} + \frac{w_b {m_c}^2}{w_b + m_c} + \frac{m_c {h_a}^2}{m_c + h_a} \ge \frac{3h_a w_b m_c}{\sqrt[3]{(h_a + w_b)(w_b + m_c)(m_c + h_a)}} \ge \frac{9h_a w_b m_c}{2(h_a + w_b + m_c)}$$
$$\ge \frac{9h_a h_b h_c}{2(m_a + m_b + m_c)} \overset{GM-HM \,\&\, Gotman}{\ge} \frac{9.\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^3}{9R} = \frac{27r^3}{R}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.