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In any Δ ABC and $\forall$ n $\in \mathbb{N}$: n $\geq$ m + 1, the following relationship holds :

$$\frac{h_a^n}{(r_a^3+h_a^3)^m}+\frac{w_b^n}{(r_b^3+w_b^3)^m}+\frac{m_c^n}{(r_c^3+m_c^3)^m}\geq\frac{2^{2m}.3^{n-3m+1}.r^n}{(9R^3-64r^3)^m}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^3+\sum_{cyc} m_a^3=\left(\sum_{cyc} r_a\right)^3-3(r_a+r_b)(r_b+r_c)(r_c+r_a)+$$

$$\left(\sum_{cyc} m_a\right)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\overset{\text{Leuenberger}}{\leq}2(4R+r)^3$$

$$-3(r_a+r_b)(r_b+r_c)(r_c+r_a)-3(h_a+h_b)(h_b+h_c)(h_c+h_a)\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Cesaro}}}{\leq}$$

$$2\left(\frac{9R}{2}\right)^3-24r_ar_br_c-24h_ah_bh_c=2\left(\frac{9R}{2}\right)^3-24rs^2-24.\frac{2r^2s^2}{R}$$

$$\overset{\substack{\text{Gerretsen + Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq}2\left(\frac{9R}{2}\right)^3-24r.27r^2-24.\frac{r^2.27Rr}{R}=2.\frac{81(9R^3-64r^3)}{8}$$

$$\Rightarrow\sum_{cyc} r_a^3+\sum_{cyc} m_a^3\leq\frac{81(9R^3-64r^3)}{4}\rightarrow(1)$$

$$\text{Also,}\left(\frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3}\right)^{\frac{m+1}{n}}\overset{\text{Power-Mean inequality}}{\geq}\left(\frac{\sum_{cyc} h_a^1}{3}\right)^1\left(\because\frac{n}{m+1}\geq 1\right)\Rightarrow\frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3}$$

$$\geq\left(\frac{\sum_{cyc} h_a}{3}\right)^{\frac{n}{m+1}}=\left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{n}{m+1}}\overset{\text{Bergstrom}}{\geq}\left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{n}{m+1}}=(3r)^{\frac{n}{m+1}}$$

$$\therefore\sum_{cyc} h_a^{\frac{n}{m+1}}\geq 3(3r)^{\frac{n}{m+1}}\rightarrow(2)$$

$$\text{Now,}\frac{h_a^n}{(r_a^3+h_a^3)^m}+\frac{w_b^n}{(r_b^3+w_b^3)^m}+\frac{m_c^n}{(r_c^3+m_c^3)^m}\geq$$

$$\frac{h_a^n}{(r_a^3+m_a^3)^m}+\frac{h_b^n}{(r_b^3+m_b^3)^m}+\frac{h_c^n}{(r_c^3+m_c^3)^m}=\sum_{cyc}\frac{\left(h_a^{\frac{n}{m+1}}\right)^{m+1}}{(r_a^3+m_a^3)^m}\overset{\text{Radon}}{\geq}$$

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$$\frac{\left(\sum_{cyc} h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left(\sum_{cyc} r_a^3+\sum_{cyc} m_a^3\right)^m} \overset{\text{via (1) and (2)}}{\geq} \frac{\left(3(3r)^{\frac{n}{m+1}}\right)^{m+1}}{\left(\frac{81(9R^3-64r^3)}{4}\right)^m}=\frac{2^{2m}.\,3^{m+1+n-4m}.\,r^n}{(9R^3-64r^3)^m}$$

$$\therefore \frac{h_a^n}{\left(r_a^3+h_a^3\right)^m}+\frac{w_b^n}{\left(r_b^3+w_b^3\right)^m}+\frac{m_c^n}{\left(r_c^3+m_c^3\right)^m} \geq \frac{2^{2m}.\,3^{n-3m+1}.\,r^n}{(9R^3-64r^3)^m}$$

$\forall \Delta ABC$," = " iff ΔABC is equilateral (QED)