ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ m, n $\in \mathbb{N}$: $n \geq m+1$, the fol$l$owing rel$a$tionship hol$d$s :

$$\frac{h_a^n}{((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2)^m}+\frac{w_b^n}{((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2)^m}$$
$$+\frac{m_c^n}{((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2)^m}\geq\frac{2^{3m}.3^{n-6m+1}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

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$$\sum_{cyc}(r_b^3+r_c^3)^2=\left(\sum_{cyc}(r_b^3+r_c^3)\right)^2-$$
$$2\left((r_b^3+r_c^3)(r_c^3+r_a^3)+(r_c^3+r_a^3)(r_a^3+r_b^3)+(r_a^3+r_b^3)(r_b^3+r_c^3)\right)\overset{A-G}{\leq}$$
$$4\left(\left(\sum_{cyc}r_a\right)^3-3(r_a+r_b)(r_b+r_c)(r_c+r_a)\right)^2$$
$$-6\sqrt[3]{(r_a^3+r_b^3)^2(r_b^3+r_c^3)^2(r_c^3+r_a^3)^2}\overset{\substack{\text{Leuenberger + Euler}\\ and\\ \text{Cesaro}}}{\leq}4\left(\left(\frac{9R}{2}\right)^3-24r_ar_br_c\right)^2$$
$$-6.\sqrt[3]{64(r_ar_br_c)^6}\overset{\text{Mitrinovic}}{\leq}4\left(\left(\frac{9R}{2}\right)^3-24.27r^3\right)^2-24(27r^3)^2$$
$$=4.\frac{81^2(9R^3-64r^3)^2}{64}-729.24r^6$$
$$\therefore\sum_{cyc}(r_b^3+r_c^3)^2\leq\frac{81^2(9R^3-64r^3)^2}{16}-729.24r^6\rightarrow(1)$$

$$\text{Again,}\sum_{cyc}(m_b^3+m_c^3)^2=\left(\sum_{cyc}(m_b^3+m_c^3)\right)^2-$$
$$2\left((m_b^3+m_c^3)(m_c^3+m_a^3)+(m_c^3+m_a^3)(m_a^3+m_b^3)+(m_a^3+m_b^3)(m_b^3+m_c^3)\right)\overset{A-G}{\leq}$$
$$4\left(\left(\sum_{cyc}m_a\right)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right)^2$$
$$-6\sqrt[3]{(m_a^3+m_b^3)^2(m_b^3+m_c^3)^2(m_c^3+m_a^3)^2}\overset{\substack{\text{Leuenberger + Euler}\\ and\\ \text{Cesaro}}}{\leq}$$
$$4\left(\left(\frac{9R}{2}\right)^3-24m_am_bm_c\right)^2-6.\sqrt[3]{64(m_am_bm_c)^6}$$

$$\le 4\left(\left(\frac{9R}{2}\right)^3 - 24h_ah_bh_c\right)^2 - 6.\sqrt[3]{64(h_ah_bh_c)^6}$$

$$= 4\left(\left(\frac{9R}{2}\right)^3 - \frac{24r^2}{R}.2s^2\right)^2 - 24\left(\frac{r^2}{R}.2s^2\right)^2 \overset{\text{Gerretsen + Euler}}{\le}$$

$$4\left(\left(\frac{9R}{2}\right)^3 - \frac{24r^2}{R}.27Rr\right)^2 - 24\left(\frac{r^2}{R}.27Rr\right)^2 = 4.\frac{81^2(9R^3-64r^3)^2}{64} - 729.24r^6$$

$$\therefore \sum_{cyc}(m_b^3+m_c^3)^2 \le \frac{81^2(9R^3-64r^3)^2}{16} - 729.24r^6 \to (2)$$

$$\text{Also}, \left(\frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3}\right)^{\frac{m+1}{n}} \overset{\text{Power-Mean inequality}}{\ge} \left(\frac{\sum_{cyc} h_a^1}{3}\right)^1 \left(\because \frac{n}{m+1} \ge 1\right)$$

$$\Rightarrow \frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3} \ge \left(\frac{\sum_{cyc} h_a}{3}\right)^{\frac{n}{m+1}} = \left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{n}{m+1}} \overset{\text{Bergstrom}}{\ge} \left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{n}{m+1}} = (3r)^{\frac{n}{m+1}}$$

$$\therefore \sum_{cyc} h_a^{\frac{n}{m+1}} \ge 3(3r)^{\frac{n}{m+1}} \to (3)$$

$$\text{Now}, \frac{h_a^n}{\left((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2\right)^m} + \frac{w_b^n}{\left((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2\right)^m}$$

$$+ \frac{m_c^n}{\left((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2\right)^m} \ge \frac{h_a^n}{\left((r_a^3+r_b^3)^2+(m_a^3+m_b^3)^2\right)^m} +$$

$$\frac{h_b^n}{\left((r_b^3+r_c^3)^2+(m_b^3+m_c^3)^2\right)^m} + \frac{h_c^n}{\left((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2\right)^m}$$

$$= \sum_{cyc} \frac{\left(h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left((r_b^3+r_c^3)^2+(m_b^3+m_c^3)^2\right)^m} \overset{\text{Radon}}{\ge} \frac{\left(\sum_{cyc} h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left(\sum_{cyc}(r_b^3+r_c^3)^2+\sum_{cyc}(m_b^3+m_c^3)^2\right)^m}$$

$$\overset{\text{via (1),(2) and (3)}}{\ge} \frac{\left(3(3r)^{\frac{n}{m+1}}\right)^{m+1}}{\left(\frac{81^2(9R^3-64r^3)^2}{16} - 729.24r^6 + \frac{81^2(9R^3-64r^3)^2}{16} - 729.24r^6\right)^m}$$

$$= \frac{3^{m+1+n}.r^n}{3^7\left(\frac{3(9R^3-64r^3)^2}{8} - 16r^6\right)^m} = \frac{2^{3m}.3^{m+1+n-7m}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

$$\therefore \frac{h_a^n}{\left((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2\right)^m} + \frac{w_b^n}{\left((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2\right)^m}$$

$$+ \frac{m_c^n}{\left((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2\right)^m} \ge \frac{2^{3m}.3^{n-6m+1}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

$\forall \Delta ABC$ *and* $\forall m, n \in \mathbb{N} : n \ge m+1$,″ =″ iff ΔABC is equi*lateral* (QED)