ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ m, n, t $\in \mathbb{N}$: $n \ge m+1$, the following relationship holds :

$$\frac{((w_a+w_b)^t+(m_a+m_b)^t)^n}{(w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5)^m}+\frac{((w_b+w_c)^t+(m_b+m_c)^t)^n}{(w_b^5+w_b^2w_c^3+w_c^5+m_b^5+m_b^2m_c^3+m_c^5)^m}+\frac{((w_c+w_a)^t+(m_c+m_a)^t)^n}{(w_c^5+w_c^2w_a^3+w_a^5+m_c^5+m_c^2m_a^3+m_a^5)^m}\ge\frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

Note:

$$(x+y+z)^5=x^5+y^5+z^5+5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$$

$$m_a+m_b+m_c\overset{Leunberger}{\le}4R+r\overset{Euler}{\le}\frac{9R}{2}$$

$$(m_a+m_b)(m_b+m_c)(m_c+m_a)\overset{AM-GM}{\ge}8m_am_bm_c$$

$$\left(m_a\ge\sqrt{s(s-a)}\right)\ge 8s^2r\overset{Mitrinovic}{\ge}8\cdot 3^3r^3$$

analog

$$\sum m_a^2+\sum m_bm_c\ge 2\sum m_bm_c$$

$$\left(\because \sum m_a^2\ge\sum m_bm_c\right)$$

$$(\because m_a\ge h_a)\ge 2\sum h_bh_c=\frac{4s^2r}{R}\ge 2\cdot 3^3\cdot r^2$$

$$\therefore \sum m_a^5=\left(\sum m_a\right)^5-5\pi(m_a+m_b)\cdot\left(\sum m_a^2+\sum m_bm_c\right)$$

$$\le\left(\frac{9R}{2}\right)^5-5\cdot 8\cdot 3^3\cdot r^3\cdot 2\cdot 3^3\cdot r^2=\frac{36(81R^5-2560r^5)}{32}$$

Applying AM-GM $h_ah_bh_c\ge 27r^3$

$$\therefore h_a+h_b+h_c\ge 3(h_ah_bh_c)^{\frac{1}{3}}=9r$$

$$w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5\le\le m_a^5+m_a^2m_b^3+m_b^5+m_a^5+m_a^2m_b^3+m_b^5$$

($w_a\le m_a$ analog)

$$=2\left(m_a^5+m_b^5\right)+m_a^2m_b^2(m_a+m_b)\le 2\left(m_a^5+m_b^5\right)+m_a^5+m_b^5=3\left(m_a^5+m_b^5\right)$$

Note:

$$\left[m_a^5+m_b^5\overset{CBS}{\ge}\frac{(m_a^4+m_b^4)(m_a+m_b)}{2}\right]\overset{AM-GM}{\ge}m_a^2m_b^2(m_a+m_b)$$

Note:

$$(w_a+w_b)^t+(m_a+m_b)^t\ge(h_a+h_b)^t+(h_a+h_b)^t\qquad\left(\because\begin{matrix}w_a\ge h_a\\ m_a\ge h_a\end{matrix}\right)$$

$$=2(h_a+h_b)^t$$

$$\therefore\left[(w_a+w_b)^t+(m_a+m_b)^t\right]^n\ge 2^n(h_a+h_b)^{tn}\overset{AM-GM}{\ge}2^{n+tn}(h_ah_b)^{\frac{tn}{2}}$$

$$\therefore LHS\ge\sum\frac{2^{n+tn}(h_ah_b)^{\frac{tn}{2}}}{3^m\left(m_a^5+m_b^5\right)^m}\overset{AM-GM}{\ge}\frac{3\cdot 2^{n+tn}}{3^m}\frac{\left[(h_ah_b)\cdot(h_bh_c)\cdot(h_ch_a)\right]^{\frac{tn}{6}}}{\left[\prod\left(m_a^5+m_b^5\right)\right]^{\frac{m}{3}}}$$

$$\overset{AM-GM}{\ge}\frac{3\cdot 2^{n+tn}}{3^m}\frac{(h_ah_bh_c)^{\frac{tn}{3}}}{\left[\frac{2\left(\sum m_a^5\right)}{3}\right]^m}\ge 3\cdot 2^{n+tn-m}\frac{(27r^3)^{\frac{tn}{3}}}{\left[\frac{36}{2^5}\left(81R^5-2560r^5\right)\right]^m}$$

$$=\frac{3\cdot 2^{n+tn-m+5m}\cdot 3^{tn}\cdot r^{tn}}{3^{6m}(81R^5-2560r^5)^m}=\frac{2^{4m+n(t+1)}\cdot 3^{nt-6m+1}\cdot r^{nt}}{(81R^5-2560r^5)^m}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0,\sum_{cyc}x^5+\sum_{cyc}x^2y^2(x+y)=\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x^2\right)\to(1)$$

$$\text{Now, }2\sum_{cyc}m_a^5+\sum_{cyc}m_a^2m_b^3$$

$$=\left(2\sum_{cyc}m_a^5+2\sum_{cyc}m_a^2m_b^3+2\sum_{cyc}m_a^3m_b^2\right)-\sum_{cyc}m_a^2m_b^3-2\sum_{cyc}m_a^3m_b^2\overset{via\,(1)}{=}$$

$$2\left(\sum_{cyc}m_a^3\right)\left(\sum_{cyc}m_a^2\right)-\sum_{cyc}m_a^3(m_b^2+m_c^2)-\sum_{cyc}m_a^3m_b^2$$

$$\overset{A-G}{\le}2\left(\sum_{cyc}m_a^3\right)\left(\sum_{cyc}m_a^2\right)-2m_am_bm_c\left(\sum_{cyc}m_a^2\right)-3m_am_bm_c.\sqrt[3]{(m_am_bm_c)^2}$$

$$=2\left(\sum_{cyc}m_a^2\right)\left(\left(\sum_{cyc}m_a\right)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)-m_am_bm_c\right)$$

$$-3m_am_bm_c.\sqrt[3]{(m_am_bm_c)^2}\overset{\substack{A-G,\\ Cesaro,\\ Leibnitz\\ and\\ Leuenberger}}{\leq} 2.\frac{3}{4}.9R^2.\left(\left(\frac{9R}{2}\right)^3-25m_am_bm_c\right)$$

$$-3m_am_bm_c.\sqrt[3]{(m_am_bm_c)^2}\leq 2.\frac{3}{4}.9R^2.\left(\left(\frac{9R}{2}\right)^3-25.27r^3\right)-3.27r^3.\sqrt[3]{(27r^3)^2}$$

$$\left(\because m_am_bm_c\geq h_ah_bh_c=\frac{r^2.2s^2}{R}\overset{Gerretsen+Euler}{\geq}\frac{r^2.27Rr}{R}=27r^3\right)$$

$$=729\left(\frac{27R^5-200R^2r^3-16r^5}{16}\right)$$

$$\therefore \boxed{2\sum_{cyc}m_a^5+\sum_{cyc}m_a^2m_b^3\leq 729\left(\frac{27R^5-200R^2r^3-16r^5}{16}\right)}\rightarrow(2)$$

$$\text{Also, }\left(\frac{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}}{3}\right)^{\frac{m+1}{tn}}\overset{Power-Mean\ inequality}{\geq}\left(\frac{\sum_{cyc}(h_a+h_b)^1}{3}\right)^1$$

$$\left(\because\frac{tn}{m+1}\geq 1\right)\Rightarrow\frac{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}}{3}\geq\left(\frac{\sum_{cyc}(h_a+h_b)}{3}\right)^{\frac{tn}{m+1}}$$

$$=2^{\frac{tn}{m+1}}.\left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{tn}{m+1}}\overset{Bergstrom}{\geq}2^{\frac{tn}{m+1}}.\left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{tn}{m+1}}=2^{\frac{tn}{m+1}}.(3r)^{\frac{tn}{m+1}}$$

$$\therefore\boxed{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}\geq 3.2^{\frac{tn}{m+1}}.(3r)^{\frac{tn}{m+1}}}\rightarrow(3)$$

$$\text{We have : }\frac{((w_a+w_b)^t+(m_a+m_b)^t)^n}{(w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5)^m}$$
$$+\frac{((w_b+w_c)^t+(m_b+m_c)^t)^n}{(w_b^5+w_b^2w_c^3+w_c^5+m_b^5+m_b^2m_c^3+m_c^5)^m}$$
$$+\frac{((w_c+w_a)^t+(m_c+m_a)^t)^n}{(w_c^5+w_c^2w_a^3+w_a^5+m_c^5+m_c^2m_a^3+m_a^5)^m}$$
$$\geq\frac{((h_a+h_b)^t+(h_a+h_b)^t)^n}{(m_a^5+m_a^2m_b^3+m_b^5+m_a^5+m_a^2m_b^3+m_b^5)^m}$$
$$+\frac{((h_b+h_c)^t+(h_b+h_c)^t)^n}{(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5)^m}$$
$$+\frac{((h_c+h_a)^t+(h_c+h_a)^t)^n}{(m_c^5+m_c^2m_a^3+m_a^5+m_c^5+m_c^2m_a^3+m_a^5)^m}$$
$$=2^n.\sum_{cyc}\frac{(h_a+h_b)^{nt}}{(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5)^m}$$

$$=2^n.\sum_{cyc}\frac{\left((h_a+h_b)^{\frac{nt}{m+1}}\right)^{m+1}}{(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5)^m}\overset{Radon}{\geq}2^n.\frac{\left(\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}\right)^{m+1}}{\left(4\sum_{cyc}m_a^5+2\sum_{cyc}m_a^2m_b^3\right)^m}$$

$$\overset{via\,(3)}{\geq} \frac{2^n.3^{m+1}.2^{tn}.(3r)^{tn}}{2^m.\left(2\sum_{cyc}m_a^5+\sum_{cyc}m_a^2m_b^3\right)^m} \overset{via\,(2)}{\geq} \frac{2^n.3^{m+1}.2^{tn}.(3r)^{tn}}{2^m.\left(729\left(\frac{27R^5-200R^2r^3-16r^5}{16}\right)\right)^m}$$

$$= \frac{2^{n+nt-m+4m}.3^{m+1+nt-6m}.r^{nt}}{(27R^5-200R^2r^3-16r^5)^m} = \frac{2^{n+nt+3m}.3^{1+nt-5m}.r^{nt}}{(27R^5-200R^2r^3-16r^5)^m}$$

$$\overset{?}{\geq} \frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$

$$\Leftrightarrow \left(\frac{3}{2}\right)^m.\frac{1}{(27R^5-200R^2r^3-16r^5)^m} \overset{?}{\geq} \frac{1}{(81R^5-2560r^5)^m}$$

$$\Leftrightarrow 3\left(81R^5-2560r^5\right) \overset{?}{\geq} 2\left(27R^5-200R^2r^3-16r^5\right)$$

$$\Leftrightarrow \boxed{189t^5+400t^2-7648 \overset{?}{\geq} 0} \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 189\left(t^5-32\right)+400\left(t^2-4\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\therefore \frac{\left((w_a+w_b)^t+(m_a+m_b)^t\right)^n}{\left(w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5\right)^m}$$

$$+\frac{\left((w_b+w_c)^t+(m_b+m_c)^t\right)^n}{\left(w_b^5+w_b^2w_c^3+w_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}$$

$$+\frac{\left((w_c+w_a)^t+(m_c+m_a)^t\right)^n}{\left(w_c^5+w_c^2w_a^3+w_a^5+m_c^5+m_c^2m_a^3+m_a^5\right)^m} \geq \frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$

$\forall \Delta ABC$ and $\forall m,n,t \in \mathbb{N} : n \geq m+1$, " = " iff ΔABC is equilateral (QED)