ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ m, n, t $\in \mathbb{N}$: $n \geq m+1$, the following relationship holds :

$$\frac{(h_a^t + w_b^t + m_c^t)^n}{(r_a^7 + 2r_a^3r_b^3(r_a + r_b) + r_b^7)^m} + \frac{(h_b^t + w_c^t + m_a^t)^n}{(r_b^7 + 2r_b^3r_c^3(r_b + r_c) + r_c^7)^m}$$
$$+ \frac{(h_c^t + w_a^t + m_b^t)^n}{(r_c^7 + 2r_c^3r_a^3(r_c + r_a) + r_a^7)^m} \geq \frac{2^{6m}.3^{n(t+1)-8m+1}.r^{nt}}{\left((27R^4 - 416r^4)(9R^3 - 64r^3)\right)^m}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\forall x,y,z > 0, \sum_{cyc} x^7 + \sum_{cyc} x^3y^3(x+y) = \left(\sum_{cyc} x^4\right)\left(\sum_{cyc} x^3\right) \to (1)$$

$$\text{Now, } \sum_{cyc} r_a^4 = \left(\sum_{cyc} r_a^2\right)^2 - 2\sum_{cyc} r_a^2r_b^2 \leq \left((4R+r)^2 - 2s^2\right)^2 - \frac{2}{3}\left(\sum_{cyc} r_ar_b\right)^2$$

$$\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\left(\frac{9R}{2}\right)^2 - 2.27r^2\right) - \frac{2}{3}.729r^4 = \frac{243}{16}\left(3\left(9R^4 + 64r^4 - 48R^2r^2\right) - 32r^4\right)$$

$$\overset{\text{Euler}}{\leq} \frac{243}{16}\left(3\left(9R^4 + 64r^4 - 48.4r^2.r^2\right) - 32r^4\right)$$

$$\therefore \sum_{cyc} r_a^4 \leq \frac{243}{16}\left(27R^4 - 416r^4\right) \to (2)$$

$$\text{Again, } \sum_{cyc} r_a^3 = (4R+r)^3 - 3(r_a+r_b)(r_b+r_c)(r_c+r_a) \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Cesaro}}}{\leq}$$

$$\left(\frac{9R}{2}\right)^3 - 24r_ar_br_c \overset{\text{Mitrinovic}}{\leq} \left(\frac{9R}{2}\right)^3 - 24.r.27r^2 \therefore \sum_{cyc} r_a^3 \leq \frac{81}{8}\left(9R^3 - 64r^3\right) \to (3)$$

$$\text{We have : } \frac{(h_a^t + w_b^t + m_c^t)^n}{(r_a^7 + 2r_a^3r_b^3(r_a + r_b) + r_b^7)^m} + \frac{(h_b^t + w_c^t + m_a^t)^n}{(r_b^7 + 2r_b^3r_c^3(r_b + r_c) + r_c^7)^m}$$

$$+ \frac{(h_c^t + w_a^t + m_b^t)^n}{(r_c^7 + 2r_c^3r_a^3(r_c + r_a) + r_a^7)^m} = \frac{\left((h_a^t + w_b^t + m_c^t)^{\frac{n}{m+1}}\right)^{m+1}}{(r_a^7 + 2r_a^3r_b^3(r_a + r_b) + r_b^7)^m} + \frac{\left((h_b^t + w_c^t + m_a^t)^{\frac{n}{m+1}}\right)^{m+1}}{(r_b^7 + 2r_b^3r_c^3(r_b + r_c) + r_c^7)^m}$$

$$+ \frac{\left((h_c^t + w_a^t + m_b^t)^{\frac{n}{m+1}}\right)^{m+1}}{(r_c^7 + 2r_c^3r_a^3(r_c + r_a) + r_a^7)^m} \overset{\text{Radon}}{\geq}$$

$$\frac{\left((h_a^t + w_b^t + m_c^t)^{\frac{n}{m+1}} + (h_b^t + w_c^t + m_a^t)^{\frac{n}{m+1}} + (h_c^t + w_a^t + m_b^t)^{\frac{n}{m+1}}\right)^{m+1}}{\left(2\sum_{cyc} r_a^7 + 2\sum_{cyc} r_a^3r_b^3(r_a + r_b)\right)^m}$$

$$\overset{\text{via }(1)}{\geq}\frac{\left(3\left(\sum_{cyc}h_a^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{2^m\cdot\left(\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}r_a^3\right)\right)^m}\overset{\text{Holder}}{\geq}\frac{3^{m+1}\cdot\left(\left(\frac{1}{3^{t-1}}\left(\sum_{cyc}h_a\right)^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{2^m\cdot\left(\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}r_a^3\right)\right)^m}$$

$$=\frac{3^{m+1}\cdot\left(\frac{1}{3^{t-1}}\left(\sum_{cyc}\frac{2rs}{a}\right)^t\right)^n}{2^m\cdot\left(\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}r_a^3\right)\right)^m}\overset{\text{Bergstrom}}{\geq}\frac{3^{m+1}\cdot\left(\frac{1}{3^{t-1}}\left(2rs\cdot\frac{9}{2s}\right)^t\right)^n}{2^m\cdot\left(\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}r_a^3\right)\right)^m}$$

$$=\frac{3^{m+1}\cdot 3^{2nt}\cdot r^{nt}}{3^{nt-n}\cdot 2^m\cdot\left(\left(\sum_{cyc}r_a^4\right)\left(\sum_{cyc}r_a^3\right)\right)^m}\overset{\text{via }(2)\text{ and }(3)}{\geq}\frac{3^{m+1+2nt-nt+n}\cdot r^{nt}}{2^m\cdot\left(\frac{243}{16}(27R^4-416r^4)\cdot\frac{81}{8}(9R^3-64r^3)\right)^m}$$

$$=\frac{3^{m+1+nt+n-9m}\cdot 2^{7m-m}\cdot r^{nt}}{\left((27R^4-416r^4)(9R^3-64r^3)\right)^m}\therefore\frac{\left(h_a^t+w_b^t+m_c^t\right)^n}{\left(r_a^7+2r_a^3r_b^3(r_a+r_b)+r_b^7\right)^m}$$

$$+\frac{\left(h_b^t+w_c^t+m_a^t\right)^n}{\left(r_b^7+2r_b^3r_c^3(r_b+r_c)+r_c^7\right)^m}+\frac{\left(h_c^t+w_a^t+m_b^t\right)^n}{\left(r_c^7+2r_c^3r_a^3(r_c+r_a)+r_a^7\right)^m}$$

$$\geq\frac{2^{6m}\cdot 3^{n(t+1)-8m+1}\cdot r^{nt}}{\left((27R^4-416r^4)(9R^3-64r^3)\right)^m}\ \forall\,\Delta ABC\text{ and }\forall\, m,n,t\in\mathbb{N},$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$