ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{24r^2}{R^2} \le \frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2} \le \frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\le \frac{(m_a+m_b)^2}{h_b^2+h_c^2}+\frac{(m_b+m_c)^2}{h_c^2+h_a^2}+\frac{(m_c+m_a)^2}{h_a^2+h_b^2}$$

$$\overset{\text{Reverse Bergstrom}}{\le} \frac{1}{4}\left(\frac{(m_a+m_b)^2+(m_c+m_a)^2}{h_b^2}\right)+\frac{1}{4}\left(\frac{(m_a+m_b)^2+(m_b+m_c)^2}{h_c^2}\right)$$

$$+\frac{1}{4}\left(\frac{(m_b+m_c)^2+(m_c+m_a)^2}{h_a^2}\right)$$

$$\le \frac{2}{4}\left(\frac{m_a^2+m_b^2+m_c^2+m_a^2}{h_b^2}\right)+\frac{2}{4}\left(\frac{m_a^2+m_b^2+m_b^2+m_c^2}{h_c^2}\right)$$

$$+\frac{2}{4}\left(\frac{m_b^2+m_c^2+m_c^2+m_a^2}{h_a^2}\right)$$

$$=\frac{1}{2}.\left(\sum_{cyc} m_a^2\right).\frac{1}{4r^2s^2}.\left(\sum_{cyc} a^2\right)$$

$$+\frac{1}{2}.\frac{1}{16r^2s^2}\left(a^2(2a^2+2b^2-c^2)+b^2(2b^2+2c^2-a^2)+c^2(2c^2+2a^2-b^2)\right)$$

$$\overset{\text{Leibnitz}}{\le} \frac{1}{2}.\frac{3}{4}.9R^2.\frac{1}{4r^2s^2}.9R^2+\frac{1}{32r^2s^2}\left(5\sum_{cyc} a^2b^2-32r^2s^2\right)\overset{\text{Mitrinovic and Goldstone}}{\le}$$

$$\frac{243R^4}{32r^2.27r^2}+\frac{5.4R^2s^2-32r^2s^2}{32r^2s^2}=\frac{9R^4+20R^2r^2-32r^4}{32r^4}\overset{?}{\le}\frac{5106R^6-64.5082r^6}{256r^6}$$

$$\Leftrightarrow 2553t^6-36t^4-80t^2-162496\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2553t^5+5106t^4+10176t^3+20352t^2+40624t+81248)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}\le\frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}$$

$$\text{Again,} \frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\ge \frac{(h_a+h_b)^2}{m_b^2+m_c^2}+\frac{(h_b+h_c)^2}{m_c^2+m_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}\overset{\text{Bergstrom}}{\ge}\frac{4\left(\sum_{cyc}\frac{2rs}{a}\right)^2}{2\sum_{cyc} m_a^2}\overset{\text{Bergstrom and Leibnitz}}{\ge}\frac{4\left(\frac{2rs.9}{2s}\right)^2}{2.\frac{3}{4}.9R^2}$$

$$=\frac{24r^2}{R^2}\therefore\frac{24r^2}{R^2}\le\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\le\frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}\ \forall\,\Delta\,\text{ABC},''=''\ \text{iff}\ \Delta\,\text{ABC is equilateral (QED)}$$