ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{144r^3}{R^2} \le \frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2} \le \frac{9(243R^7-30976r^7)}{32r^6}$$

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$$\frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$
$$\le \frac{(m_a+m_b)^3}{h_b^2+h_c^2}+\frac{(m_b+m_c)^3}{h_c^2+h_a^2}+\frac{(m_c+m_a)^3}{h_a^2+h_b^2}$$
$$\overset{\text{Reverse Bergstrom}}{\le} \frac{1}{4}\left(\frac{(m_a+m_b)^3+(m_c+m_a)^3}{h_b^2}+\frac{(m_a+m_b)^3+(m_b+m_c)^3}{h_c^2}+\frac{(m_b+m_c)^3+(m_c+m_a)^3}{h_a^2}\right)$$
$$\overset{\text{Holder}}{\le} \frac{4}{4}\left(\frac{m_a^3+m_b^3+m_c^3+m_a^3}{h_b^2}+\frac{m_a^3+m_b^3+m_b^3+m_c^3}{h_c^2}+\frac{m_b^3+m_c^3+m_c^3+m_a^3}{h_a^2}\right)$$
$$=\frac{2\sum_{cyc}m_a^3-(m_b^3+m_c^3)}{h_b^2}+\frac{2\sum_{cyc}m_a^3-(m_c^3+m_a^3)}{h_c^2}+\frac{2\sum_{cyc}m_a^3-(m_a^3+m_b^3)}{h_a^2}$$
$$=2\left(\sum_{cyc}m_a^3\right)\left(\frac{\sum_{cyc}a^2}{4r^2s^2}\right)-\left(\frac{m_b^3}{h_b^2}+\frac{m_c^3}{h_b^2}+\frac{m_c^3}{h_c^2}+\frac{m_a^3}{h_c^2}+\frac{m_a^3}{h_a^2}+\frac{m_b^3}{h_a^2}\right)$$
$$\overset{\substack{\text{Radon}\\ \text{and}\\ \text{Leibnitz}}}{\le} 2\left((4R+r)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right)\left(\frac{9R^2}{4r^2s^2}\right)-\frac{\left(2\sum_{cyc}m_a\right)^3}{\left(2\sum_{cyc}h_a\right)^2}$$
$$\overset{\substack{\text{Euler,}\\ \text{Mitrinovic}\\ \text{and}\\ \text{Cesaro}}}{\le} 2\left(\left(\frac{9R}{2}\right)^3-24h_ah_bh_c\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-2\sum_{cyc}\frac{2rs}{a}$$
$$\overset{\text{Bergstrom}}{\le} 2\left(\left(\frac{9R}{2}\right)^3-24.\frac{2r^2s^2}{R}\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-2.\frac{2rs.9}{2s}$$
$$\overset{\text{Gerretsen + Euler}}{\le} 2\left(\left(\frac{9R}{2}\right)^3-24.\frac{r^2.27Rr}{R}\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-18r$$
$$=9\left(\frac{3R^2(9R^3-64r^3)-32r^5}{16r^4}\right)\overset{?}{\le}\frac{9(243R^7-30976r^7)}{32r^6}$$
$$\Leftrightarrow 243t^7-54t^5+384t^2-30912\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(243t^6+486t^5+918t^4+1836t^3+3672t^2+7728t+15456\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2} \leq \frac{9\left(243R^7-30976r^7\right)}{32r^6}$$

$$\text{Again,} \frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$

$$\geq \frac{(h_a+h_b)^3}{m_b^2+m_c^2}+\frac{(h_b+h_c)^3}{m_c^2+m_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2} \overset{\text{Holder}}{\geq} \frac{8\left(\sum_{\text{cyc}}\frac{2rs}{a}\right)^3}{3.2\sum_{\text{cyc}} m_a^2} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\geq} \frac{8\left(\frac{2rs.9}{2s}\right)^3}{3.2.\frac{3}{4}.9R^2}$$

$$= \frac{144r^3}{R^2} \therefore \frac{144r^3}{R^2} \leq \frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$

$$\leq \frac{9\left(243R^7-30976r^7\right)}{32r^6}\ \forall \Delta ABC, '' = ''\ \text{iff } \Delta ABC \text{ is equilateral (QED)}$$