ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{\left(h_a^3+h_a^2w_b+w_b^3\right)^5}{(w_a^3+w_a^2m_b+m_b^3)^3}+\frac{\left(h_b^3+h_b^2w_c+w_c^3\right)^5}{(w_b^3+w_b^2m_c+m_c^3)^3}+\frac{(h_c^3+h_c^2w_a+w_a^3)^5}{(w_c^3+w_c^2m_a+m_a^3)^3}\geq\frac{6^9\cdot r^{15}}{(9R^3-64r^3)^3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\left(h_a^3+h_a^2w_b+w_b^3\right)^5}{\left(w_a^3+w_a^2m_b+m_b^3\right)^3}+\frac{\left(h_b^3+h_b^2w_c+w_c^3\right)^5}{\left(w_b^3+w_b^2m_c+m_c^3\right)^3}+\frac{\left(h_c^3+h_c^2w_a+w_a^3\right)^5}{\left(w_c^3+w_c^2m_a+m_a^3\right)^3}$$

$$\geq\sum_{cyc}\frac{\left(h_a^3+h_a^2h_b+h_b^3\right)^5}{\left(m_a^3+m_a^2m_b+m_b^3\right)^3}\overset{Panaitopol}{\geq}\sum_{cyc}\frac{\left(h_a^3+h_a^2h_b+h_b^3\right)^5}{\left(\frac{R^9}{512r^9}*\left(h_a^3+h_a^2h_b+h_b^3\right)\right)^3}$$

$$\geq\frac{512r^9}{R^9}\sum_{cyc}\left(h_ah_b(h_a+h_b)+h_a^2h_b\right)^2=\frac{512r^9}{R^9}h_a^2h_b^2h_c^2\sum_{cyc}\left(\frac{2h_a+h_b}{h_c}\right)^2\geq$$

$$\frac{512r^9}{R^9}*\frac{4r^4s^4}{R^2}*\frac{1}{3}\left(2\sum_{cyc}\frac{h_b}{h_a}+\sum_{cyc}\frac{h_a}{h_b}\right)^2\overset{Gerretsen}{\geq}$$

$$\frac{512r^9}{R^9}*r^4\left(\frac{27Rr+5r(R-2r)}{R}\right)^2*\frac{1}{3}\left(2\sum_{cyc}\frac{h_b}{h_a}+\sum_{cyc}\frac{h_a}{h_b}\right)^2\overset{\substack{Euler\\and\\A-G}}{\geq}$$

$$\frac{512r^9}{R^9}*r^4*729r^2*\frac{1}{3}*(6+3)^2=\frac{6^9*r^{15}}{R^9}\overset{?}{\geq}\frac{6^9*r^{15}}{(9R^3-64r^3)^3}\Leftrightarrow 9R^3-64r^3\overset{?}{\geq}R^3$$

$$\Leftrightarrow 8R^3\overset{?}{\geq}64r^3\Leftrightarrow R\overset{?}{\geq}2r\rightarrow\text{true via Euler}\therefore\frac{\left(h_a^3+h_a^2w_b+w_b^3\right)^5}{\left(w_a^3+w_a^2m_b+m_b^3\right)^3}+$$

$$\frac{\left(h_b^3+h_b^2w_c+w_c^3\right)^5}{\left(w_b^3+w_b^2m_c+m_c^3\right)^3}+\frac{\left(h_c^3+h_c^2w_a+w_a^3\right)^5}{\left(w_c^3+w_c^2m_a+m_a^3\right)^3}\geq\frac{6^9*r^{15}}{(9R^3-64r^3)^3}$$

$\forall\,\Delta$ ABC," = " iff Δ ABC is equilateral (QED)