ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{576r^4}{9R^3-64r^3} \le \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3} \le \frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$

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$$\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$
$$\le \frac{(m_a+m_b)^4}{h_b^3+h_c^3}+\frac{(m_b+m_c)^4}{h_c^3+h_a^3}+\frac{(m_c+m_a)^4}{h_a^3+h_b^3}$$
$$\overset{\text{Reverse Bergstrom}}{\le} \frac{1}{4}\left(\frac{(m_a+m_b)^4+(m_c+m_a)^4}{h_b^3}\right)+\frac{1}{4}\left(\frac{(m_a+m_b)^4+(m_b+m_c)^4}{h_c^3}\right)$$
$$+\frac{1}{4}\left(\frac{(m_b+m_c)^4+(m_c+m_a)^4}{h_a^3}\right)$$
$$\overset{\text{Holder}}{\le} \frac{8}{4}\left(\frac{m_a^4+m_b^4+m_c^4+m_a^4}{h_b^3}+\frac{m_a^4+m_b^4+m_b^4+m_c^4}{h_c^3}+\frac{m_b^4+m_c^4+m_c^4+m_a^4}{h_a^3}\right)$$
$$=2\left(\frac{2\sum_{cyc}m_a^4-(m_b^4+m_c^4)}{h_b^3}+\frac{2\sum_{cyc}m_a^4-(m_c^4+m_a^4)}{h_c^3}+\frac{2\sum_{cyc}m_a^4-(m_a^4+m_b^4)}{h_a^3}\right)$$
$$=4\left(\sum_{cyc}m_a^4\right)\left(\frac{\sum_{cyc}a^3}{8r^3s^3}\right)-2\left(\frac{m_b^4}{h_b^3}+\frac{m_c^4}{h_b^3}+\frac{m_c^4}{h_c^3}+\frac{m_a^4}{h_c^3}+\frac{m_a^4}{h_a^3}+\frac{m_b^4}{h_a^3}\right)$$
$$\overset{\text{Radon}}{\le} 4.\frac{9}{16}.\left(\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2\right).\frac{2s(s^2-6Rr-3r^2)}{8r^3s^3}-2.\frac{\left(2\sum_{cyc}m_a\right)^4}{\left(2\sum_{cyc}h_a\right)^3}$$
$$\overset{\substack{\text{Leibnitz,}\\\text{Gordon,}\\\text{Mitrinovic}\\\text{and}\\\text{Gerretsen}}}{\le} 4.\frac{9}{16}.\left(81R^4-\frac{2}{3}.48r^2.27r^2\right).\frac{4R^2-2Rr}{4r^3.27r^2}-4.\sum_{cyc}\frac{2rs}{a}\overset{\text{Bergstrom}}{\le}$$
$$\frac{9(81R^4-864r^4)(2R^2-Rr)}{8r^3.27r^2}-4.\frac{2rs.9}{2s}=9.\frac{(3R^4-32r^4)(2R^2-Rr)-32r^6}{8r^5}$$
$$\overset{?}{\le}\frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$
$$\Leftrightarrow 19683t^{10}-96t^6+48t^5+1024t^2-512t-20153856\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$
$$(t-2)\begin{pmatrix}19683t^9+39366t^8+78732t^7+157464t^6+314832t^5\\+629712t^4+1259424t^3+2518848t^2+5038720t+10076928\end{pmatrix}\overset{?}{\ge}0$$
$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2 \therefore \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$
$$\le\frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$

$$\text{Again}, \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\geq \frac{(h_a+h_b)^4}{m_b^3+m_c^3}+\frac{(h_b+h_c)^4}{m_c^3+m_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\overset{\text{Holder}}{\geq} \frac{16\left(\sum_{cyc}\frac{2rs}{a}\right)^4}{9.2\sum_{cyc}m_a^3} \overset{\substack{\text{Bergstrom,}\\ \text{Leuenberger}\\ \text{and}\\ \text{Cesaro}}}{\geq} \frac{8\left(\frac{2rs.9}{2s}\right)^4}{9((4R+r)^3-24m_am_bm_c)} \overset{\text{Euler}}{\geq} \frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24h_ah_bh_c}$$

$$= \frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24.\frac{2r^2s^2}{R}} \overset{\text{Gerretsen + Euler}}{\geq} \frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24.\frac{r^2.27Rr}{R}} = \frac{576r^4}{9R^3-64r^3}$$

$$\therefore \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3} \geq \frac{576r^4}{9R^3-64r^3}$$

$$\therefore \frac{576r^4}{9R^3-64r^3} \leq \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\leq \frac{9\left(19683R^{10}-20154368r^{10}\right)}{128r^9}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$