ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^3(r_a^2+r_br_c)}{r_b^4+r_c^4}+\frac{r_b^3(r_b^2+r_cr_a)}{r_c^4+r_a^4}+\frac{r_c^3(r_c^2+r_ar_b)}{r_a^4+r_b^4}\ge 9r$$

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WLOG: $a\le b\le c$.

$$-a\ge -b\ge -c\Rightarrow s-a\ge s-b\ge s-c\Rightarrow \frac{1}{s-a}\le\frac{1}{s-b}\le\frac{1}{s-c}$$

$$\frac{F}{s-a}\le\frac{F}{s-b}\le\frac{F}{s-c}\Rightarrow r_a\le r_b\le r_c$$

$$\sum_{cyc}\frac{r_a^3(r_a^2+r_br_c)}{r_b^4+r_c^4}\ge\sum_{cyc}\frac{r_a^3(r_a^2+r_ar_a)}{r_b^4+r_c^4}=\sum_{cyc}\frac{r_a^5+r_a^5}{r_b^4+r_c^4}\ge$$

$$\overset{LEHMER}{\ge}\sum_{cyc}\frac{r_a^4+r_a^4}{r_b^3+r_c^3}\overset{LEHMER}{\ge}\sum_{cyc}\frac{r_a^3+r_a^3}{r_b^2+r_c^2}\overset{LEHMER}{\ge}\sum_{cyc}\frac{r_a^2+r_a^2}{r_b+r_c}\overset{LEHMER}{\ge}$$

$$\ge\sum_{cyc}\frac{r_b+r_c}{1+1}=\sum_{cyc}r_a\overset{AM-GM}{\ge}3\sqrt[3]{r_ar_br_c}=$$

$$=3\sqrt[3]{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}=3\sqrt[3]{\frac{sF^3}{s(s-a)(s-b)(s-c)}}=$$

$$=3\sqrt[3]{\frac{sF^3}{F^2}}=3\sqrt[3]{sF}=3\sqrt[3]{rs^2}\overset{MITRINOVIC}{\ge}3\sqrt[3]{r\left(3\sqrt{3}r\right)^2}=$$

$$=3\sqrt[3]{27r^3}=3\cdot 3r=9r$$

Equality holds for: $a=b=c$.