ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{h_a(w_b^2+m_c^2)}{h_a^2+w_bm_c}+\frac{w_b(m_c^2+h_a^2)}{w_b^2+m_ch_a}+\frac{m_c(h_a^2+w_b^2)}{m_c^2+h_aw_b}\geq 9r$$

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Firstly, we shall prove that $\forall\, x,y,z>0$, $\prod_{cyc}(y^2+z^2)\geq\prod_{cyc}(x^2+yz)$

$$\Leftrightarrow \sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4\overset{(i)}{\geq} xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3$$

$$\text{LHS of (i)}=\sum_{cyc}\frac{x^4y^2+x^4z^2}{2}+\sum_{cyc}\frac{x^4y^2+x^2y^4}{2}\overset{A-G}{\geq}\sum_{cyc}x^4yz+\sum_{cyc}x^3y^3$$

$$=xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3\Rightarrow \text{(i) is true} \therefore \forall\, x,y,z>0, \frac{\prod_{cyc}(y^2+z^2)}{\prod_{cyc}(x^2+yz)}\geq 1\rightarrow (1)$$

$$\text{Now, }\frac{h_a(w_b^2+m_c^2)}{h_a^2+w_bm_c}+\frac{w_b(m_c^2+h_a^2)}{w_b^2+m_ch_a}+\frac{m_c(h_a^2+w_b^2)}{m_c^2+h_aw_b}$$

$$\overset{A-G}{\geq} 3\sqrt[3]{h_aw_bm_c.\frac{(w_b^2+m_c^2)(m_c^2+h_a^2)(h_a^2+w_b^2)}{(h_a^2+w_bm_c)(w_b^2+m_ch_a)(m_c^2+h_aw_b)}}$$

$$\geq 3.\sqrt[3]{h_ah_bh_c.\frac{(y^2+z^2)(z^2+x^2)(x^2+y^2)}{(x^2+yz)(y^2+zx)(z^2+xy)}}\overset{\text{via }(1)}{\geq} 3.\sqrt[3]{\frac{2r^2s^2}{R}}$$

$$\overset{\text{Gerretsen}}{\geq} 3.\sqrt[3]{\frac{r^2.(27Rr+5r(R-2r))}{R}}\overset{\text{Euler}}{\geq} 3.\sqrt[3]{\frac{r^2.27Rr}{R}}=9r$$

$$\therefore \frac{h_a(w_b^2+m_c^2)}{h_a^2+w_bm_c}+\frac{w_b(m_c^2+h_a^2)}{w_b^2+m_ch_a}+\frac{m_c(h_a^2+w_b^2)}{m_c^2+h_aw_b}\geq 9r\ \forall\,\Delta\text{ ABC},$$

'' ='' iff Δ ABC is equilateral (QED)