ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, the following relationship holds :

$$\frac{h_a^2(h_a^2+w_bm_c)}{(w_b+m_c)^2}+\frac{w_b^2(w_b^2+m_ch_a)}{(m_c+h_a)^2}+\frac{m_c^2(m_c^2+h_aw_b)}{(h_a+w_b)^2}\ge\frac{27r^2}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $h_a = x, w_b = y$ and $m_c = z$ and then :

$$\frac{h_a^2(h_a^2+w_bm_c)}{(w_b+m_c)^2}+\frac{w_b^2(w_b^2+m_ch_a)}{(m_c+h_a)^2}+\frac{m_c^2(m_c^2+h_aw_b)}{(h_a+w_b)^2}$$

$$=\sum_{cyc}\frac{x^2(x^2+yz)}{(y+z)^2}=\sum_{cyc}\frac{x^4}{(y+z)^2}+xyz\sum_{cyc}\frac{x}{(y+z)^2}$$

$$\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}\frac{x^2}{(y+z)^2}\right)+\frac{xyz}{3}\left(\sum_{cyc}\frac{1}{y+z}\right)\left(\sum_{cyc}\frac{x}{y+z}\right)$$

$$\left(\because \text{WLOG assuming } x\ge y\ge z\Rightarrow x^2\ge y^2\ge z^2, \frac{x^2}{(y+z)^2}\ge\frac{y^2}{(z+x)^2}\ge\frac{z^2}{(x+y)^2}, \frac{1}{y+z}\ge\frac{1}{z+x}\ge\frac{1}{x+y} \text{ and } \frac{x}{y+z}\ge\frac{y}{z+x}\ge\frac{z}{x+y}\right)$$

$$\overset{\text{Nesbitt and Bergstrom}}{\ge}\frac{1}{3}\left(\sum_{cyc}x^2\right)\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)^2+\frac{xyz}{3}\cdot\frac{9}{2\sum_{cyc}x}\cdot\frac{3}{2}\overset{\text{Nesbitt}}{\ge}$$

$$\frac{1}{9}\left(\sum_{cyc}x^2\right)\cdot\frac{9}{4}+\frac{9xyz}{4\sum_{cyc}x}\overset{?}{\ge}\frac{3}{2}\cdot\sqrt[3]{x^2y^2z^2}\Leftrightarrow \boxed{\frac{\sum_{cyc}x^2}{2}+\frac{9xyz}{2\sum_{cyc}x}\underset{(*)}{\overset{?}{\ge}}3.\sqrt[3]{x^2y^2z^2}}$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(1)\Rightarrow x=s-X, y=s-Y,$ $z=s-Z$ and such substitutions $\Rightarrow xyz=(s-X)(s-Y)(s-Z)$

$$\Rightarrow xyz=r^2s\to(2);\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(3)$$

$$\text{and }\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (3)}}{=}s^2-2(4Rr+r^2)$$

$$\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\to(4)\therefore\text{ via (1),(2) and (4)},(*)\Leftrightarrow$$

$$\frac{s^2-8Rr-2r^2}{2}+\frac{9r^2s}{2s}\ge3.\sqrt[3]{r^4s^2}\Leftrightarrow\boxed{(s^2-8Rr+7r^2)^3-216r^4s^2\overset{(**)}{\ge}0}$$

$$and \because \left(s^2-16Rr+5r^2\right)^3 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\left(s^2-8Rr+7r^2\right)^3-216r^4s^2 \geq \left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow \left(24Rr+6r^2\right)s^4-r^2s^2\left(576R^2-144Rr+144r^2\right)$$

$$+r^3\left(3584R^3-2496R^2r+24Rr^2+218r^3\right) \overset{(***)}{\geq} 0 \text{ and}$$

$$\because \left(24Rr+6r^2\right)\left(s^2-16Rr+5r^2\right)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (***) \geq \left(24Rr+6r^2\right)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow \boxed{\left(48R^2+24Rr-51r^2\right)s^2 \overset{(****)}{\geq} r\left(640R^3+48R^2r-96Rr^2-17r^3\right)}$$

$$\text{Now, LHS of } (****) \overset{Gerretsen}{\geq} \left(48R^2+24Rr-51r^2\right)\left(16Rr-5r^2\right) \overset{?}{\geq}$$

$$r\left(640R^3+48R^2r-96Rr^2-17r^3\right) \Leftrightarrow 16t^3+12t^2-105t+34 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(16t^2+44t-17\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**)$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{h_a^2\left(h_a^2+w_bm_c\right)}{(w_b+m_c)^2}+\frac{w_b^2\left(w_b^2+m_ch_a\right)}{(m_c+h_a)^2}+\frac{m_c^2\left(m_c^2+h_aw_b\right)}{(h_a+w_b)^2}$$

$$\geq \frac{3}{2}\cdot\sqrt[3]{h_a^2w_b^2m_c^2} \geq \frac{3}{2}\cdot\sqrt[3]{h_a^2h_b^2h_c^2}=\frac{3}{2},\left(\sqrt[3]{\frac{2r^2s^2}{R}}\right)^2 \overset{Gerretsen}{\geq}$$

$$\frac{3}{2}\cdot\left(\sqrt[3]{\frac{r^2.\left(27Rr+5r(R-2r)\right)}{R}}\right)^2 \overset{Euler}{\geq} \frac{3}{2}\cdot\left(\sqrt[3]{\frac{r^2.27Rr}{R}}\right)^2$$

$$\therefore \frac{h_a^2\left(h_a^2+w_bm_c\right)}{(w_b+m_c)^2}+\frac{w_b^2\left(w_b^2+m_ch_a\right)}{(m_c+h_a)^2}+\frac{m_c^2\left(m_c^2+h_aw_b\right)}{(h_a+w_b)^2} \geq \frac{27r^2}{2},$$

$$\forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x, y, z > 0$. We have

$$\sum_{cyc}\frac{x^2(x^2+yz)}{(y+z)^2}$$

$$=\sum_{cyc}\frac{(x^2)^3}{(xy+zx)^2}+xyz.\sum_{cyc}\frac{x^3}{(xy+zx)^2} \overset{Hölder}{\geq} \frac{\left(\sum_{cyc}x^2\right)^3}{4\left(\sum_{cyc}yz\right)^2}$$

$$+xyz.\frac{\left(\sum_{cyc}x\right)^3}{4\left(\sum_{cyc}yz\right)^2}$$

$$\underset{\left(\sum_{cyc}x\right)^2 \geq 3\sum_{cyc}yz}{\overset{\sum_{cyc}x^2 \geq \sum_{cyc}yz}{\geq}} \frac{\sum_{cyc}yz}{4}+\frac{3xyz\sum_{cyc}x}{4\sum_{cyc}yz} \overset{AM-GM}{\geq} \frac{1}{2}\sqrt{3xyz(x+y+z)}.$$

Setting $x = h_a,\ \ y = w_b,\ \ z = m_c$, we obtain

$$\frac{h_a^2(h_a^2+w_bm_c)}{(w_b+m_c)^2}+\frac{w_b^2(w_b^2+m_ch_a)}{(m_c+h_a)^2}+\frac{m_c^2(m_c^2+h_aw_b)}{(h_a+w_b)^2}$$
$$\geq\frac{1}{2}\sqrt{3h_aw_bm_c(h_a+w_b+m_c)}$$
$$\underset{m_c\geq h_c}{\overset{w_b\geq h_b}{\geq}}\frac{3}{2}\sqrt{h_ah_bh_c.\frac{h_a+h_b+h_c}{3}}\underset{AM-HM}{\overset{GM-HM}{\geq}}\frac{3}{2}\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2=\frac{3}{2}\left(\frac{3}{\frac{1}{r}}\right)^2=\frac{27r^2}{2},$$

as desired. Equality holds iff ΔABC is equilateral.