ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^n(r_a^2+r_br_c)}{r_b^{n+1}+r_c^{n+1}}+\frac{r_b^n(r_b^2+r_cr_a)}{r_c^{n+1}+r_a^{n+1}}+\frac{r_c^n(r_c^2+r_ar_b)}{r_a^{n+1}+r_b^{n+1}}\geq 9r, n\in\mathbb{N}$$

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WLOG: $a\leq b\leq c$.

$$-a\geq -b\geq -c\Rightarrow s-a\geq s-b\geq s-c\Rightarrow \frac{1}{s-a}\leq\frac{1}{s-b}\leq\frac{1}{s-c}$$

$$\frac{F}{s-a}\leq\frac{F}{s-b}\leq\frac{F}{s-c}\Rightarrow r_a\leq r_b\leq r_c$$

$$\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{r_b^{n+1}+r_c^{n+1}}\geq\sum_{cyc}\frac{r_a^n(r_a^2+r_ar_a)}{r_b^{n+1}+r_c^{n+1}}=\sum_{cyc}\frac{r_a^{n+2}+r_a^{n+2}}{r_b^{n+1}+r_c^{n+1}}\geq$$

$$\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^{n+1}+r_a^{n+1}}{r_b^n+r_c^n}\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^n+r_a^n}{r_b^{n-1}+r_c^{n-1}}\overset{LEHMER}{\geq}\dots\geq\sum_{cyc}\frac{r_a^2+r_a^2}{r_b+r_c}\overset{LEHMER}{\geq}$$

$$\geq\sum_{cyc}\frac{r_b+r_c}{1+1}=\sum_{cyc}r_a\overset{AM-GM}{\geq}3\sqrt[3]{r_ar_br_c}=$$

$$=3\sqrt[3]{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}=3\sqrt[3]{\frac{sF^3}{s(s-a)(s-b)(s-c)}}=$$

$$=3\sqrt[3]{\frac{sF^3}{F^2}}=3\sqrt[3]{sF}=3\sqrt[3]{rs^2}\overset{MITRINOVIC}{\geq}3\sqrt[3]{r\left(3\sqrt{3}r\right)^2}=$$

$$=3\sqrt[3]{27r^3}=3\cdot 3r=9r$$

Equality holds for: $a=b=c$.