ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^2+r_br_c}{r_a^4(r_b+r_c)}+\frac{r_b^2+r_cr_a}{r_b^4(r_c+r_a)}+\frac{r_c^2+r_ar_b}{r_c^4(r_a+r_b)}\ge\frac{16r}{9R^4}$$

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WLOG: $a\le b\le c$.

$$-a\ge -b\ge -c\Rightarrow s-a\ge s-b\ge s-c\Rightarrow\frac{1}{s-a}\le\frac{1}{s-b}\le\frac{1}{s-c}$$

$$\frac{F}{s-a}\le\frac{F}{s-b}\le\frac{F}{s-c}\Rightarrow r_a\le r_b\le r_c$$

$$\sum_{cyc}\frac{r_a^2+r_br_c}{r_a^4(r_b+r_c)}\ge\sum_{cyc}\frac{r_a^2+r_ar_a}{r_a^4(r_b+r_c)}=\sum_{cyc}\frac{2r_a^2}{r_a^4(r_b+r_c)}=$$

$$=2\sum_{cyc}\frac{\frac{1}{r_a^2}}{r_b+r_c}\overset{BERGSTROM}{\ge}2\cdot\frac{\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^2}{2(r_a+r_b+r_c)}=$$

$$=\frac{\frac{1}{r^2}}{4R+r}=\frac{r}{r^3(4R+r)}\overset{EULER}{\ge}\frac{r}{(\frac{R}{2})^3\left(4R+\frac{R}{2}\right)}=\frac{16r}{9R^4}$$

Equality holds for: $a=b=c$.