ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ n $\in \mathbb{N}$, the following relationship holds :

$$\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}$$

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$\boxed{\text{Case 1}}$ n = 1 and then : $\text{LHS}=\frac{x^2+yz}{x(y+z)}+\frac{y^2+zx}{y(z+x)}+\frac{z^2+xy}{z(x+y)}$

$$(x=h_a,y=w_b,z=m_c)=\sum_{\text{cyc}}\frac{x}{y+z}+\frac{1}{xyz}\sum_{\text{cyc}}\frac{y^2z^2}{y+z}\overset{\substack{\text{Nesbitt}\\\text{and}\\\text{Bergstrom}}}{\ge}\frac{3}{2}+\frac{\left(\sum_{\text{cyc}}xy\right)^2}{2xyz\sum_{\text{cyc}}x}$$

$$=\frac{3}{2}+\frac{3}{2}=3\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad(\because n=1)$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\text{for } n=1$$

$\boxed{\text{Case 2}}$ n = 2 and then : $\boxed{\text{LHS}}=\frac{x^2+yz}{x^2(y+z)}+\frac{y^2+zx}{y^2(z+x)}+\frac{z^2+xy}{z^2(x+y)}=\sum_{\text{cyc}}\frac{1}{y+z}$

$$+xyz\sum_{\text{cyc}}\frac{\left(\frac{1}{x}\right)^3}{y+z}\overset{\substack{\text{Bergstrom}\\\text{and}\\\text{Holder}}}{\ge}\frac{9}{2\sum_{\text{cyc}}x}+xyz.\frac{\left(\sum_{\text{cyc}}\frac{1}{x}\right)^3}{6\sum_{\text{cyc}}x}\boxed{\overset{?}{\ge}\frac{9}{\sum_{\text{cyc}}x}}$$

$$\Leftrightarrow xyz.\frac{\left(\sum_{\text{cyc}}\frac{1}{x}\right)^3}{6\sum_{\text{cyc}}x}\overset{?}{\ge}\frac{9}{2\sum_{\text{cyc}}x}\Leftrightarrow\left(\sum_{\text{cyc}}xy\right)^3\overset{?}{\ge}27x^2y^2z^2\rightarrow\text{true via AM}-\text{GM}$$

$$\therefore \text{LHS}\ge\frac{9}{\sum_{\text{cyc}}x}=\frac{9}{h_a+w_b+m_c}\ge\frac{9}{\sum_{\text{cyc}}m_a}\overset{\substack{\text{Leuenberger}\\\text{and}\\\text{Euler}}}{\ge}\frac{9}{\frac{9R}{2}}=\frac{2}{R}=\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}$$

$$(n=2)\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\text{for } n=2$$

$\boxed{\text{Case 3}}$ n = 3 and then : $\boxed{\text{LHS}}=\frac{x^2+yz}{x^3(y+z)}+\frac{y^2+zx}{y^3(z+x)}+\frac{z^2+xy}{z^3(x+y)}=\sum_{\text{cyc}}\frac{1}{xy+xz}$

$$+xyz\sum_{\text{cyc}}\frac{\left(\frac{1}{x}\right)^4}{y+z}\overset{\substack{\text{Bergstrom}\\\text{and}\\\text{Holder}}}{\ge}\frac{9}{2\sum_{\text{cyc}}xy}+xyz.\frac{\left(\sum_{\text{cyc}}\frac{1}{x}\right)^4}{9.2\sum_{\text{cyc}}x}=\frac{9}{2\sum_{\text{cyc}}xy}+\frac{\left(\sum_{\text{cyc}}xy\right)^4}{18x^3y^3z^3\sum_{\text{cyc}}x}$$

$$\ge\frac{9}{2\sum_{\text{cyc}}xy}+\frac{9x^2y^2z^2\left(\sum_{\text{cyc}}x\right)^2}{18x^3y^3z^3\sum_{\text{cyc}}x}=\frac{9}{2\sum_{\text{cyc}}xy}+\frac{\sum_{\text{cyc}}x}{2xyz}\boxed{\overset{?}{\ge}\frac{1}{3}\left(\frac{9}{\sum_{\text{cyc}}x}\right)^2}$$

$$\Leftrightarrow 9xyz\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^3\overset{?}{\geq}54xyz\sum_{cyc}xy\to \text{true}$$

$$\because 9xyz\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^3\overset{A-G}{\geq}9xyz.3\sum_{cyc}xy+\left(\sum_{cyc}xy\right).27xyz$$

$$=54xyz\sum_{cyc}xy\therefore \text{LHS}\geq\frac{1}{3}\left(\frac{9}{\sum_{cyc}x}\right)^2=\frac{1}{3}\left(\frac{9}{h_a+w_b+m_c}\right)^2\geq\frac{1}{3}\left(\frac{9}{\sum_{cyc}m_a}\right)^2$$

$$\overset{\substack{\text{Leuenberger}\\ and\\ \text{Euler}}}{\geq}\frac{1}{3}\left(\frac{9}{\frac{9R}{2}}\right)^2=\frac{1}{3}\left(\frac{2}{R}\right)^2=\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad(\because n=3)$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\geq\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad \text{for } n=3$$

$$\boxed{Case\ 4}\ n\in\mathbb{N}-\{1,2,3\}\ and\ \text{then}:\boxed{\text{LHS}}=\frac{x^2+yz}{x^n(y+z)}+\frac{y^2+zx}{y^n(z+x)}+\frac{z^2+xy}{z^n(x+y)}$$

$$=\sum_{cyc}\frac{\left(\frac{1}{x}\right)^{n-2}}{y+z}+xyz.\sum_{cyc}\frac{\left(\frac{1}{x}\right)^{n+1}}{y+z}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}}{3^{n-4}.2\sum_{cyc}x}+xyz.\frac{\left(\sum_{cyc}\frac{1}{x}\right)^{n+1}}{3^{n-1}.2\sum_{cyc}x}$$

$$\boxed{\overset{?}{\geq}\frac{1}{3^{n-2}}\cdot\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}\cdot\frac{9}{\sum_{cyc}x}}\Leftrightarrow\frac{9}{2\sum_{cyc}x}+\frac{xyz\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\overset{?}{\geq}\frac{9}{\sum_{cyc}x}$$

$$\Leftrightarrow\frac{xyz\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\overset{?}{\geq}\frac{9}{2\sum_{cyc}x}\Leftrightarrow\left(\sum_{cyc}xy\right)^3\overset{?}{\geq}27x^2y^2z^2\to\text{true via AM}-\text{GM}$$

$$\therefore\text{LHS}\geq\frac{1}{3^{n-2}}\cdot\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}\cdot\frac{9}{\sum_{cyc}x}\overset{\text{Bergstrom}}{\geq}\frac{1}{3^{n-2}}\cdot\left(\frac{9}{\sum_{cyc}x}\right)^{n-2}\cdot\frac{9}{\sum_{cyc}x}$$

$$=\frac{1}{3^{n-2}}\cdot\left(\frac{9}{\sum_{cyc}x}\right)^{n-1}=\frac{1}{3^{n-2}}\cdot\left(\frac{9}{h_a+w_b+m_c}\right)^{n-1}\geq\frac{1}{3^{n-2}}\cdot\left(\frac{9}{\sum_{cyc}m_a}\right)^{n-1}$$

$$\overset{\substack{\text{Leuenberger}\\ and\\ \text{Euler}}}{\geq}\frac{1}{3^{n-2}}\cdot\left(\frac{9}{\frac{9R}{2}}\right)^{n-1}$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\geq\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\forall\, n\in\mathbb{N}-\{1,2,3\}$$

$$\therefore \text{combining } all\ cases,\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}$$

$$\geq\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\forall\,\Delta\,\text{ABC}\ and\ \forall\, n\in\mathbb{N},''=''\ \text{ iff }\Delta\,\text{ABC is equi}lateral\ (\text{QED})$$