ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\ge\frac{16}{27R^4}$$

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$$\forall\, x,y>0,\frac{x^5+y^5}{x^4y^4(x+y)}=\frac{(x+y)\left(\left(x^4+x^2y^2+y^4\right)-\left(x^3y+xy^3\right)\right)}{x^4y^4(x+y)}=$$

$$\frac{\left(x^2+y^2\right)^2-x^2y^2-xy\left(x^2+y^2\right)}{x^4y^4}=\frac{\left(x^2+y^2+xy\right)\left(x^2+y^2-xy\right)-xy\left(x^2+y^2\right)}{x^4y^4}$$

$$\overset{A-G}{\ge}\frac{xy\left(x^2+y^2+xy\right)-xy\left(x^2+y^2\right)}{x^4y^4}=\frac{1}{x^2y^2}\Rightarrow\frac{x^5+y^5}{x^4y^4(x+y)}\ge\frac{1}{x^2y^2}\to(1)$$

$$\text{and analogously},\frac{y^5+z^5}{y^4z^4(y+z)}\ge\frac{1}{y^2z^2}\to(2)\text{ and }\frac{z^5+x^5}{z^4x^4(z+x)}\ge\frac{1}{z^2x^2}\to(3)$$

and via $x\equiv m_a, y\equiv m_b, z\equiv m_c$, we arrive at :

$$\frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\ge\sum_{cyc}\frac{1}{m_a^2m_b^2}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{9}{\sum_{cyc}m_a^2m_b^2}=\frac{9}{\frac{9}{16}\sum_{cyc}a^2b^2}\overset{\text{Goldstone}}{\ge}\frac{16}{4R^2s^2}\overset{\text{Mitrinovic}}{\ge}\frac{16}{4R^2s^2}=\frac{16}{R^2.27R^2}$$

$$\therefore\frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\ge\frac{16}{27R^4}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)