ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\frac{16r}{9R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x^5+y^5}{x^3y^3(x^2+y^2)}=\frac{(x^2+y^2)(x^3+y^3)-x^2y^2(x+y)}{x^3y^3(x^2+y^2)}$$

$$=\frac{x^3+y^3}{x^3y^3}-\frac{x+y}{xy(x^2+y^2)}\ge\frac{xy(x+y)}{x^3y^3}-\frac{2(x+y)}{xy(x+y)^2}=\frac{x+y}{x^2y^2}-\frac{2}{xy(x+y)}$$

$$=\frac{(x+y)^2-2xy}{x^2y^2(x+y)}=\frac{x^2+y^2}{x^2y^2(x+y)}\ge\frac{(x+y)^2}{2x^2y^2(x+y)}\Rightarrow\frac{x^5+y^5}{x^3y^3(x^2+y^2)}\ge\frac{x+y}{2x^2y^2}$$

$\rightarrow$ (1) and analogously, $\frac{y^5+z^5}{y^3z^3(y^2+z^2)}\ge\frac{y+z}{2y^2z^2}\rightarrow$ (2) and $\frac{z^5+x^5}{z^3x^3(z^2+x^2)}\ge\frac{z+x}{2z^2x^2}$

$\rightarrow$ (3) and via $x\equiv r_a, y\equiv r_b, z\equiv r_c$, we arrive at :

$$\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\sum_{cyc}\frac{r_a+r_b}{2r_a^2r_b^2}$$

$$=\frac{1}{2r_a^2r_b^2r_c^2}\cdot\sum_{cyc}r_c^2(r_a+r_b)=\frac{1}{2r^2s^4}\cdot\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-3r_ar_br_c\right)$$

$$=\frac{(4R+r)s^2-3rs^2}{2r^2s^4}=\frac{2R-r}{r^2s^2}\overset{\substack{\text{Euler}\\\text{and}\\\text{Mitrinovic}}}{\ge}\frac{3r}{\frac{R^2}{4}\cdot\frac{27R^2}{4}}$$

$$\therefore\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\frac{16r}{9R^4}\ \forall\,\Delta\text{ ABC},$$

'' = '' iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}\overset{Chebyshev}{\ge}\sum_{cyc}\frac{(r_b^3+r_c^3)(r_b^2+r_c^2)}{2r_b^3r_c^3(r_b^2+r_c^2)}=$$

$$=\frac{1}{2}\sum_{cyc}\left(\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\sum_{cyc}\frac{1}{r_a^3}$$

$$\overset{Hölder}{\geq} \frac{1}{3^2}\left(\sum_{cyc}\frac{1}{r_a}\right)^3 = \frac{1}{9r^3} \overset{Euler}{\geq} \frac{1}{9r^3}\cdot\left(\frac{2r}{R}\right)^4 = \frac{16r}{9R^4},$$

as desired. Equality holds iff ΔABC is equilateral.