In any Δ ABC, the following relationship holds :

$$\frac{6r}{R}\le\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)$$

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$$\frac{1}{am_a}\sum_{cyc}a^2\ge 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\ge\frac{12}{\left(\sum_{cyc}a^2\right)^2}\Leftrightarrow$$

$$\left(\sum_{cyc}a^2\right)^2-3a^2\left(2b^2+2c^2-a^2\right)\ge 0\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2\left(2\sum_{cyc}a^2-3a^2\right)\ge 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-6a^2\sum_{cyc}a^2+9a^4\ge 0\Leftrightarrow\left(\sum_{cyc}a^2-3a^2\right)^2\ge 0$$

$$\Leftrightarrow\left(b^2+c^2-2a^2\right)^2\ge 0\to\text{true}\Rightarrow m_a\le\frac{\sum_{cyc}a^2}{2\sqrt{3}a}\text{ and analogs}\to(1)$$

$$\text{Now, }\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}=\frac{bm_a+cm_b+am_c}{2rs}$$

$$\overset{\substack{\text{via (1)}\\\text{and}\\\text{Mitrinovic}}}{\le}\frac{\sum_{cyc}a^2}{2\sqrt{3}}\cdot\frac{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}{2r.3\sqrt{3}r}\overset{\substack{\text{Leibnitz}\\\text{and}\\\text{CBS}}}{\le}\frac{9R^2}{36r^2}\cdot\sqrt{\sum_{cyc}a^2}\cdot\sqrt{\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}\overset{\substack{\text{Leibnitz}\\\text{and}\\\text{Goldstone}}}{\le}\frac{R^2}{4r^2}\cdot\sqrt{\frac{9R^2.4R^2s^2}{16R^2r^2s^2}}$$

$$=\frac{3R^3}{8r^3}=\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-8\left(\frac{R}{r}\right)^3\right)\overset{\text{Euler}}{\le}\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-8.8\right)$$

$$\therefore\boxed{\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)}$$

$$\text{Again, }\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\ge\frac{h_a}{m_b}+\frac{h_b}{m_c}+\frac{h_c}{m_a}=\frac{h_a^2}{m_bh_a}+\frac{h_b^2}{h_bm_c}+\frac{h_c^2}{m_ah_c}$$

$$\ge\frac{h_a^2}{m_bm_a}+\frac{h_b^2}{m_bm_c}+\frac{h_c^2}{m_am_c}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}h_a\right)^2}{\sum_{cyc}m_bm_c}=\frac{\left(\sum_{cyc}ab\right)^2}{4R^2.\frac{\left(\sum_{cyc}m_a\right)^2-\sum_{cyc}m_a^2}{2}}$$

$$\overset{\text{Chu and Yang}}{\ge}\frac{\left(s^2+4Rr+r^2\right)^2}{4R^2.\frac{4s^2-16Rr+5r^2-\frac{3}{2}(s^2-4Rr-r^2)}{2}}=\frac{\left(s^2+4Rr+r^2\right)^2}{R^2\left(5s^2-20Rr+13r^2\right)}\overset{?}{\ge}\frac{6r}{R}$$

$$\Leftrightarrow s^4-\left(22Rr-2r^2\right)s^2+r^2\left(136R^2-70Rr+r^2\right)\underset{(*)}{\overset{?}{\ge}}0\text{ and}$$

$$\because\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen}}{\ge}0\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$\text{LHS of }(*)\ge\left(s^2-16Rr+5r^2\right)^2\Leftrightarrow(5R-4r)s^2\overset{(**)}{\ge}r\left(60R^2-45Rr+12r^2\right)$$

$$\text{Now, } (5R-4r)s^2 \overset{\text{Gerretsen}}{\geq} (5R-4r)(16Rr-5r^2) \overset{?}{\geq} r(60R^2-45Rr+12r^2)$$

$$\Leftrightarrow 4(5R^2-11Rr+2r^2) \overset{?}{\geq} 0 \Leftrightarrow 4(5R-r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a} \geq \frac{6r}{R}} \therefore \frac{6r}{R} \leq \frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}$$

$$\leq \frac{3}{8}\left(9\left(\frac{R}{r}\right)^3 - 64\right) \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$