ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\frac{64\cdot r^5}{(9R^3-64r^3)^2}$$

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WLOG we may assume $a\ge b\ge c$ and then : $r_a\ge r_b\ge r_c;\ m_a\le m_b\le m_c;$

$$\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}\Rightarrow\frac{r_a^5}{m_a^4}\ge\frac{r_b^5}{m_b^4}\ge\frac{r_c^5}{m_c^4}\ \text{and}\ \frac{1}{h_a^2}\ge\frac{1}{h_b^2}\ge\frac{1}{h_c^2}\to(1)$$

$$\text{Now, }\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\sum_{cyc}\left(\frac{1}{h_a^2}*\frac{r_a^5}{m_a^4}\right)\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}\frac{1}{h_a^2}\right)\left(\sum_{cyc}\frac{r_a^5}{m_a^4}\right)\overset{\text{Radon}}{\ge}\frac{1}{3*4r^2s^2}\left(\sum_{cyc}a^2\right)\frac{\left(\sum_{cyc}r_a\right)^5}{\left(\sum_{cyc}m_a\right)^4}\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{Ionescu-Weitzenbock + Mitrinovic}}}{\ge}$$

$$\frac{1}{3*4r^2s^2}*\left(4\sqrt{3}r*3\sqrt{3}r\right)*\frac{\left(\sum_{cyc}r_a\right)^5}{\left(\sum_{cyc}r_a\right)^4}=\frac{3(4R+r)}{s^2}$$

$$\overset{\text{Mitrinovic}}{\ge}\frac{3(4R+r)}{\frac{27R^2}{4}}\overset{?}{\ge}\frac{64*r^5}{(9R^3-64r^3)^2}$$

$$\Leftrightarrow 324t^7+81t^6-4608t^4-1152t^3-144t^2+16384t+4096\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\begin{pmatrix}324t^5+1377t^4+4212t^3\\+6732t^2+8928t+8640\end{pmatrix}+15232\right)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\therefore\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\frac{64*r^5}{(9R^3-64r^3)^2}$$

$\forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)