ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\frac{4r}{R^2} \leq \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \leq \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right)$$

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$$\frac{1}{am_a}\sum_{cyc} a^2 \geq 2\sqrt{3} \Leftrightarrow \frac{1}{a^2 m_a^2} \geq \frac{12}{\left(\sum_{cyc} a^2\right)^2} \Leftrightarrow$$

$$\left(\sum_{cyc} a^2\right)^2 - 3a^2(2b^2 + 2c^2 - a^2) \geq 0 \Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 3a^2\left(2\sum_{cyc} a^2 - 3a^2\right) \geq 0$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 6a^2\sum_{cyc} a^2 + 9a^4 \geq 0 \Leftrightarrow \left(\sum_{cyc} a^2 - 3a^2\right)^2 \geq 0$$

$$\Leftrightarrow (b^2 + c^2 - 2a^2)^2 \geq 0 \to \text{true} \Rightarrow m_a \leq \frac{\sum_{cyc} a^2}{2\sqrt{3}a} \text{ and analogs} \to (1)$$

$$\text{Now, } \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \leq \frac{m_a}{h_b h_c} + \frac{m_b}{h_c h_a} + \frac{m_c}{h_a h_b} \overset{\text{via (1)}}{\leq} \frac{\sum_{cyc} a^2}{2\sqrt{3}} \cdot \left(\sum_{cyc} \frac{4R^2}{a.\,ca.\,ab}\right)$$

$$\leq \frac{\sum_{cyc} a^2}{2\sqrt{3}} \cdot \frac{4R^2}{4Rrs} \cdot \sum_{cyc} \frac{1}{4(s-b)(s-c)} = \frac{\sum_{cyc} a^2}{2\sqrt{3}} \cdot \frac{R}{rs} \cdot \frac{s}{4r^2 s} \overset{\substack{\text{Leibnitz} \\ \text{and} \\ \text{Mitrinovic}}}{\leq} \frac{9R^3}{8\sqrt{3}r^3 .\, 3\sqrt{3}r}$$

$$= \frac{R^3}{8r^4} \overset{?}{\leq} \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right) = \frac{1}{2r} \cdot \left(\frac{81R^5 - 2560r^5}{16r^5}\right)$$

$$\Leftrightarrow 81t^5 - 4t^3 - 2560 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(81t^4 + 162t^3 + 320t^2 + 640t + 1280) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \leq \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right)$$

$$\text{Again, } \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \geq \frac{h_a}{m_b m_c} + \frac{h_b}{m_c m_a} + \frac{h_c}{m_a m_b} = \sum_{cyc} \frac{h_a^2}{h_a m_b m_c}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^2}{\sum_{cyc} h_a m_b m_c} \overset{\text{Chebyshev}}{\geq} \frac{3\left(\sum_{cyc} h_a\right)^2}{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} m_b m_c\right)}$$

$$(\because \text{WLOG assuming } a \geq b \geq c \Rightarrow h_a \leq h_b \leq h_c \text{ and } m_b m_c \geq m_c m_a \geq m_a m_b)$$

$$= \frac{3\left(\sum_{cyc} h_a\right)}{\frac{\left(\sum_{cyc} m_a\right)^2 - \sum_{cyc} m_a^2}{2}} \overset{\text{Chu and Yang}}{\geq} \frac{3(s^2 + 4Rr + r^2)}{2R.\frac{4s^2 - 16Rr + 5r^2 - \frac{3}{2}(s^2 - 4Rr - r^2)}{2}}$$

$$= \frac{6(s^2 + 4Rr + r^2)}{R(5s^2 - 20Rr + 13r^2)} \overset{?}{\geq} \frac{4r}{R^2} \Leftrightarrow (3R - 10r)s^2 + r(12R^2 + 43Rr - 26r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

$\boxed{Case\ 1}$ $3R - 10r \geq 0$ *and* then : LHS of $(*) \geq r(12R^2 + 43Rr - 26r^2) \overset{\text{Euler}}{\geq}$ $r(12R^2 + 86r^2 - 26r^2) > 0 \Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $3R - 10r < 0$ *and* then : LHS of $(*) = -(10r - 3R)s^2 + r\left(\begin{matrix} 12R^2 + \\ 43Rr - 26r^2 \end{matrix}\right)$

$$\overset{\text{Gerretsen}}{\geq} -(10r - 3R)(4R^2 + 4Rr + 3r^2) + r(12R^2 + 43Rr - 26r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^3 - 4t^2 + 3t - 14 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(t-2)(3t^2 + 2t + 7) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall \Delta ABC$

$$\therefore \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \geq \frac{4r}{R^2} \text{ and so,}$$

$$\frac{4r}{R^2} \leq \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \leq \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right)$$

$\forall \Delta ABC$, "=" iff $\Delta ABC$ is equilateral (QED)