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In any Δ ABC, the following relationship holds :

$$\frac{18r^2}{R} \le \frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \le \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2 - 2r^2\right)$$

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$$\frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \le \frac{m_a m_b}{h_c} + \frac{m_b m_c}{h_a} + \frac{m_c m_a}{h_b} = m_a m_b m_c . \sum_{cyc} \frac{1}{h_a m_a}$$

$$\overset{m_a m_b m_c \le \frac{Rs^2}{2}}{\le} \frac{Rs^2}{2} . \sum_{cyc} \frac{1}{h_a^2} = \frac{Rs^2}{2} . \frac{1}{4r^2 s^2} . \sum_{cyc} a^2 \overset{\text{Leibnitz}}{\le} \frac{Rs^2}{2} . \frac{9R^2}{4r^2 s^2} = \frac{9R^3}{8r^2}$$

$$\overset{?}{\le} \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2 - 2r^2\right) \Leftrightarrow 3R^4 - 32r^4 \overset{?}{\ge} 2R^3 r \Leftrightarrow 3t^4 - 2t^3 - 32 \overset{?}{\ge} 0 \; \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t^3 + 4t^2 + 8t + 16) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \le \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2 - 2r^2\right)$$

$$\text{Again}, \frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \ge \frac{h_a h_b}{m_c} + \frac{h_b h_c}{m_a} + \frac{h_c h_a}{m_b} = \sum_{cyc} \frac{ca.ab}{4R^2 m_a}$$

$$= \frac{4Rrs}{4R^2} . \sum_{cyc} \frac{a^2}{a m_a} \overset{\text{Bergstrom}}{\ge} \frac{rs}{R} . \frac{4s^2}{\sum_{cyc} a m_a} \overset{\text{Chebyshev}}{\ge} \frac{rs}{R} . \frac{4s^2}{\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} m_a\right)}$$

$$(\because \text{WLOG assuming } a \ge b \ge c \Rightarrow m_a \le m_b \le m_c) \overset{\text{Leuenberger}}{\ge} \frac{12rs.s^2}{2Rs(4R+r)}$$

$$\overset{\text{Euler}}{\ge} \frac{6rs^2}{\frac{9R^2}{2}} = \frac{2r.2s^2}{3R^2} \overset{\text{Gerretsen}}{\ge} \frac{2r.\left(27Rr + 5r(R-2r)\right)}{3R^2} \overset{\text{Euler}}{\ge} \frac{2r.(27Rr)}{3R^2}$$

$$\therefore \frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \ge \frac{18r^2}{R}$$

$$\therefore \frac{18r^2}{R} \le \frac{m_a m_b}{m_c} + \frac{w_b w_c}{w_a} + \frac{h_c h_a}{h_b} \le \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2 - 2r^2\right)$$

$\forall \, \Delta$ ABC," = " iff Δ ABC is equilateral (QED)

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$$

$$-r^3(4R+r)^3\overset{?}{\le}0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underbrace{\overset{?}{\ge}}_{(\bullet\bullet)}20rs^4$$

$$\text{Now, LHS of }(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)}s^2(16Rr-5r^2)(8R-16r)$$

$$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\ \textit{and}$$

$$\text{RHS of }(\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(b)}20rs^2(4R^2+4Rr+3r^2)$$

$(a),(b)\Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$

$$\ge 20rs^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\ge}27r^2s^2$$

$$\text{Now, LHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\textit{and}\text{ RHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(d)}27r^2(4R^2+4Rr+3r^2)$$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\ge 27r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\ge 0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\ge 0\rightarrow\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$

$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\le\frac{Rs^2}{2}\ \text{(QED)}$$