ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{48r^3}{28R^3-192r^3} \le \frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \le 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)$$

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$$\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \le \frac{m_a}{h_b+h_c}+\frac{m_b}{h_c+h_a}+\frac{m_c}{h_a+h_b}$$

$$\overset{\text{Reverse Bergstrom}}{\le} \frac{1}{4}\sum_{cyc}\left(\frac{m_a}{h_b}+\frac{m_a}{h_c}\right) \overset{\text{CBS}}{\le} \frac{1}{4}.\sqrt{2\sum_{cyc} m_a^2}.\sqrt{2\sum_{cyc}\frac{1}{h_a^2}}$$

$$= \frac{1}{4}.\sqrt{4.\frac{3}{4}.\sum_{cyc} a^2.\frac{1}{4r^2s^2}.\sum_{cyc} a^2} \overset{\text{Leibnitz and Mitrinovic}}{\le} \frac{1}{4}.\sqrt{\frac{3.81R^4}{4r^2.27r^2}} = \sqrt{\frac{9R^4}{64r^4}}$$

$$\overset{?}{\le} 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right) = \sqrt{\frac{9(7R^3-48r^3)^2}{256r^6}} \Leftrightarrow (7t^3-48)^2 \overset{?}{\ge} 4t^4 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 49t^6-4t^4-672t^3+2304 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(49t^4+196t^3+584t^2+880t+1184)+1216\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \boxed{\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \le 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)}$$

$$\text{Again}, \frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \ge \frac{h_a}{m_b+m_c}+\frac{h_b}{m_c+m_a}+\frac{h_c}{m_a+m_b}$$

$$\overset{\text{Chebyshev}}{\ge} \frac{1}{3}\left(\sum_{cyc} h_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)\left(\begin{matrix}\because \text{WLOG assuming } a \ge b \ge c \Rightarrow h_a \le h_b \le h_c \text{ and} \\ \frac{1}{m_b+m_c} \le \frac{1}{m_c+m_a} \le \frac{1}{m_a+m_b}\end{matrix}\right)$$

$$= \frac{1}{3}\left(2rs\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right) \overset{\text{Bergstrom}}{\ge} \frac{1}{3}.2rs.\frac{9}{2s}.\frac{9}{2\sum_{cyc} m_a} \overset{\text{Leuenberger + Euler}}{\ge} 3r.\frac{9}{2.\frac{9R}{2}}$$

$$= \frac{3r}{R} \overset{?}{\ge} \frac{48r^3}{28R^3-192r^3} \Leftrightarrow 28t^3-16t-192 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(28t^2+56t+96) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \boxed{\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \ge \frac{48r^3}{28R^3-192r^3}}$$

$$\text{and so}, \frac{48r^3}{28R^3-192r^3} \le \frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b} \le 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)$$

$\forall \Delta$ ABC, "=" iff Δ ABC is equilateral (QED)