ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{48r^3}{R^3} \le \frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \le \frac{3}{4}\cdot\left(9.\left(\frac{R}{r}\right)^3 - 64\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \le \frac{m_a + m_b}{h_c} + \frac{m_b + m_c}{h_a} + \frac{m_c + m_a}{h_b}$$

$$= \frac{m_a}{h_c} + \frac{m_b}{h_c} + \frac{m_b}{h_a} + \frac{m_c}{h_a} + \frac{m_c}{h_b} + \frac{m_a}{h_b} \overset{CBS}{\le} \sqrt{2\sum_{cyc} m_a^2}\cdot\sqrt{2\sum_{cyc}\frac{1}{h_a^2}}$$

$$= \sqrt{4.\frac{3}{4}.\sum_{cyc} a^2 . \frac{1}{4r^2s^2}.\sum_{cyc} a^2} \overset{\substack{\text{Leibnitz}\\ and\\ \text{Mitrinovic}}}{\le} \sqrt{\frac{3.81R^4}{4r^2.27r^2}} = \frac{3R^2}{2r^2} \overset{?}{\le} \frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3 - 64\right)$$

$$= \frac{3(9R^3 - 64r^3)}{4r^3} \Leftrightarrow 9t^3 - 64 \overset{?}{\ge} 2t^2 \left(t = \frac{R}{r}\right) \Leftrightarrow 9t^3 - 2t^2 - 64 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)(9t^2 + 16t + 32) \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \le \frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3 - 64\right)$$

$$\text{Again, } \frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \ge \frac{h_a + h_b}{m_c} + \frac{h_b + h_c}{m_a} + \frac{h_c + h_a}{m_b} \overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}(h_b + h_c)\right)\left(\sum_{cyc}\frac{1}{m_a}\right)\left(\begin{array}{c}\because \text{WLOG assuming } a \ge b \ge c \Rightarrow \\ h_b + h_c \ge h_c + h_a \ge h_a + h_b \text{ and } \frac{1}{m_a} \ge \frac{1}{m_b} \ge \frac{1}{m_c}\end{array}\right)$$

$$= \frac{2}{3}\left(2rs\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{m_a}\right) \overset{\text{Bergstrom}}{\ge} \frac{2}{3}.2rs.\frac{9}{2s}.\frac{9}{\sum_{cyc} m_a} \overset{\text{Leuenberger + Euler}}{\ge} 6r.\frac{9}{\frac{9R}{2}}$$

$$= \frac{12r}{R} = \frac{12R^2r}{R^3} \overset{\text{Euler}}{\ge} \frac{12r.4r^2}{R^3} \therefore \frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \ge \frac{48r^3}{R^3} \text{ and so,}$$

$$\frac{48r^3}{R^3} \le \frac{h_a + h_b}{h_c} + \frac{w_b + w_c}{w_a} + \frac{m_c + m_a}{m_b} \le \frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3 - 64\right)$$

$\forall \Delta$ ABC," = " iff Δ ABC is equilateral (QED)