ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{96r^3}{28R^3-192r^3}\le\frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le 168.\left(\frac{R}{4r}\right)^3-18$$

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$$\frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le\frac{m_a+m_b}{h_b+h_c}+\frac{m_b+m_c}{h_c+h_a}+\frac{m_c+m_a}{h_a+h_b}$$

$$\overset{\text{Reverse Bergstrom}}{\le}\frac{1}{4}\left(\frac{m_a+m_b}{h_b}+\frac{m_a+m_b}{h_c}\right)+\frac{1}{4}\left(\frac{m_b+m_c}{h_c}+\frac{m_b+m_c}{h_a}\right)$$

$$+\frac{1}{4}\left(\frac{m_c+m_a}{h_a}+\frac{m_c+m_a}{h_b}\right)=\frac{1}{4}\left(\frac{\sum_{cyc}m_a+m_c}{h_a}+\frac{\sum_{cyc}m_a+m_a}{h_b}+\frac{\sum_{cyc}m_a+m_b}{h_c}\right)$$

$$=\frac{1}{4}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right)+\frac{1}{4}\left(\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\right)\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{CBS}}}{\le}$$

$$\frac{4R+r}{4r}+\frac{1}{4}.\sqrt{\sum_{cyc}m_a^2.\sum_{cyc}\frac{1}{h_a^2}}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le}\frac{4R+r}{4r}+\frac{1}{4}.\sqrt{\frac{3}{4}.\frac{9R^2.9R^2}{4r^2.27r^2}}$$

$$=\frac{3R^2+4r(4R+r)}{16r^2}\overset{?}{\le}168.\left(\frac{R}{4r}\right)^3-18=\frac{21R^3-144r^3}{8r^3}$$

$$\Leftrightarrow 42t^3-3t^2-16t-292\overset{?}{\ge}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(42t^2+81t+146)\to\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\ \therefore\ \frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le 168.\left(\frac{R}{4r}\right)^3-18$$

$$\text{Again,}\ \frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\ge\frac{h_a+h_b}{m_b+m_c}+\frac{h_b+h_c}{m_c+m_a}+\frac{h_c+h_a}{m_a+m_b}$$

$$=2rs\left(\begin{array}{c}\frac{1}{am_b+am_c}+\frac{1}{bm_b+bm_c}+\frac{1}{bm_c+bm_a}+\frac{1}{cm_c+cm_a}\\ +\frac{1}{cm_a+cm_b}+\frac{1}{am_a+am_b}\end{array}\right)\overset{\text{Bergstrom}}{\ge}$$

$$\frac{2rs.36}{2(am_b+bm_c+cm_a)+(am_a+bm_b+cm_c)+(bm_a+cm_b+am_c)}$$

$$\overset{\text{CBS}}{\ge}\frac{72rs}{2.\sqrt{\sum_{cyc}m_a^2.\sum_{cyc}a^2}+\sqrt{\sum_{cyc}m_a^2.\sum_{cyc}a^2}+\sqrt{\sum_{cyc}m_a^2.\sum_{cyc}a^2}}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\ge}\frac{18r.3\sqrt{3}r}{\sqrt{\frac{3}{4}.9R^2.9R^2}}$$

$$=\frac{12r^2}{R^2}\overset{?}{\ge}\frac{96r^3}{28R^3-192r^3}\Leftrightarrow 7t^3-2t^2-48\overset{?}{\ge}0\Leftrightarrow(t-2)(7t^2+12t+24)\to\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\ \therefore\ \frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\ge\frac{96r^3}{28R^3-192r^3}\ \text{and so,}$$

$$\frac{96r^3}{28R^3-192r^3}\le\frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le 168.\left(\frac{R}{4r}\right)^3-18$$

$\forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)