ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$3.\left(\frac{2r}{R}\right)^6 \le \frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a} \le 3.\left(\frac{R}{2r}\right)^6$$

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$$\frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a} \overset{A-G}{\ge} 3.\sqrt[3]{\frac{m_a m_b}{w_a w_b}.\frac{w_b w_c}{h_b h_c}.\frac{h_c h_a}{m_c m_a}} = 3.\sqrt[3]{\frac{m_b w_c h_a}{w_a h_b m_c}}$$

$$\ge 3.\sqrt[3]{\frac{h_b h_c h_a}{m_a m_b m_c}} \overset{m_a m_b m_c \le \frac{Rs^2}{2}}{\ge} 3.\sqrt[3]{\frac{\frac{2r^2 s^2}{R}}{\frac{Rs^2}{2}}} = 3.\sqrt[3]{\frac{4r^2.2r}{R^2.2r}} \overset{Euler}{\ge} 3.\sqrt[3]{\frac{8r^3}{R^3}} = 3.\left(\frac{2r}{R}\right)$$

$$= \frac{3.\left(\frac{2r}{R}\right)^6}{\left(\frac{2r}{R}\right)^5} \overset{Euler}{\ge} 3.\left(\frac{2r}{R}\right)^6 \therefore \frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a} \ge 3.\left(\frac{2r}{R}\right)^6$$

$$\text{Again,} \frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a} \le \frac{m_a m_b}{h_a h_b} + \frac{m_b m_c}{h_b h_c} + \frac{m_c m_a}{h_c h_a}$$

$$\overset{Panaitopol}{\le} \left(\frac{R}{2r}\right)^2.\sum_{cyc}\frac{h_b h_c}{h_b h_c} = \frac{3.\left(\frac{R}{2r}\right)^6}{\left(\frac{R}{2r}\right)^4} \overset{Euler}{\le} 3.\left(\frac{R}{2r}\right)^6 \therefore \frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a}$$

$$\le 3.\left(\frac{R}{2r}\right)^6 \text{ and so, } 3.\left(\frac{2r}{R}\right)^6 \le \frac{m_a m_b}{w_a w_b} + \frac{w_b w_c}{h_b h_c} + \frac{h_c h_a}{m_c m_a} \le 3.\left(\frac{R}{2r}\right)^6$$

$\forall\, \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Proof of $m_a m_b m_c \le \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4 b^2 + \sum_{cyc} a^2 b^4\right) + 3a^2 b^2 c^2\right\}$$

$$\text{Now,} \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\ \therefore (1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(\begin{array}{c}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{array}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{array}{c}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{array}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R-16r)+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of }(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2\left(16Rr-5r^2\right)(8R-16r)$$

$$+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of }(\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2\left(4R^2+4Rr+3r^2\right)$$

$(a),(b) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$$s^2\left(16Rr-5r^2\right)(8R-16r)+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3$$

$$\geq 20rs^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow s^2\left(108R^2-256Rr+53r^2\right)+r(4R+r)^3 \geq 0$$

$$\Leftrightarrow s^2\left(108R^2-256Rr+80r^2\right)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3$$

$$\text{and RHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2\left(4R^2+4Rr+3r^2\right)$$

$(c),(d) \Rightarrow$ **in order to prove** $(\bullet\bullet\bullet)$, **it suffices to prove :**

$$\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3 \geq 27r^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(224t+309)+648\right) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$