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In any Δ ABC, the following relationship holds :

$$\frac{192r^4}{3R^4-32r^4}\le\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2$$

$$\le 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\le$$

$$2\left(\frac{m_a^2}{m_b^2}+\frac{w_b^2}{w_a^2}+\frac{w_b^2}{w_c^2}+\frac{h_c^2}{h_b^2}+\frac{h_c^2}{h_a^2}+\frac{m_a^2}{m_c^2}\right)\le 2\left(\frac{m_a^2}{h_b^2}+\frac{m_b^2}{h_a^2}+\frac{m_b^2}{h_c^2}+\frac{m_c^2}{h_b^2}+\frac{m_c^2}{h_a^2}+\frac{m_a^2}{h_c^2}\right)$$

$$=2\sum_{cyc}\frac{m_b^2+m_c^2}{h_a^2}=2\sum_{cyc}\frac{\sum_{cyc}m_a^2-m_a^2}{h_a^2}=\frac{2}{4r^2s^2}\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}a^2\right)-2\sum_{cyc}\frac{m_a^2}{h_a^2}$$

$$\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le}\frac{\frac{3}{4}.81R^4}{2r^2.27r^2}-6\ \therefore\ \left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\overset{(1)}{\le}\frac{9R^4}{8r^4}-6$$

$$\text{Again, }9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24\overset{\text{Euler}}{\ge}9.\left(\frac{243}{32}.16.\left(\frac{R}{r}\right)^2-484\right)^2-24$$

$$\overset{?}{\ge}\frac{9R^4}{8r^4}-6\Leftrightarrow\frac{\left(243t^2-968\right)^2}{4}\overset{?}{\ge}\frac{t^4}{8}+2=\frac{t^4+16}{8}$$

$$\Leftrightarrow 118097t^4-940896t^2+1874032\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(t+2)\big((t-2)(118097t+236194)+3880\big)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\therefore 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24\ge\frac{9R^4}{8r^4}-6$$

$$\overset{\text{via (1)}}{\ge}\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2$$

$$\therefore\boxed{\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\le 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24}$$

$$\text{Also, }\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\overset{A-G}{\ge}4\left(\frac{m_a}{m_b}.\frac{w_b}{w_a}+\frac{w_b}{w_c}.\frac{h_c}{h_b}+\frac{h_c}{h_a}.\frac{m_a}{m_c}\right)$$

$$\overset{A-G}{\ge}12\sqrt[3]{\frac{m_a^2w_b^2h_c^2}{(m_bm_c)(w_cw_a)(h_ah_b)}}=12\sqrt[3]{\frac{m_a^3w_b^3h_c^3}{\left(\prod_{cyc}m_a\right)\left(\prod_{cyc}w_a\right)\left(\prod_{cyc}h_a\right)}}$$

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$$\geq 12\left(\prod_{cyc} h_a\right)\sqrt[3]{\frac{1}{\left(\prod_{cyc} m_a\right)^3}} \geq 12\frac{\left(\prod_{cyc} h_a\right)}{\left(\prod_{cyc} m_a\right)} \overset{m_a m_b m_c \leq \frac{Rs^2}{2}}{\geq} 12.\frac{\frac{2r^2s^2}{R}}{\frac{Rs^2}{2}} = \frac{48r^2}{R^2}$$

$$\overset{?}{\geq} \frac{192r^4}{3R^4 - 32r^4} \Leftrightarrow 3t^4 - 4t^2 - 32 \overset{?}{\geq} 0 \Leftrightarrow (t-2)\left(3t^3 + 6t^2 + 8t + 16\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \boxed{\left(\frac{m_a}{m_b} + \frac{w_b}{w_a}\right)^2 + \left(\frac{w_b}{w_c} + \frac{h_c}{h_b}\right)^2 + \left(\frac{h_c}{h_a} + \frac{m_a}{m_c}\right)^2 \geq \frac{192r^4}{3R^4 - 32r^4}}$$

$$\text{and so,} \frac{192r^4}{3R^4 - 32r^4} \leq \left(\frac{m_a}{m_b} + \frac{w_b}{w_a}\right)^2 + \left(\frac{w_b}{w_c} + \frac{h_c}{h_b}\right)^2 + \left(\frac{h_c}{h_a} + \frac{m_a}{m_c}\right)^2$$

$$\leq 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6 - 484\right)^2 - 24 \; \forall \, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$