ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{64r^5}{R^6} \le \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{m_a}{h_b^2} + \frac{m_b}{h_c^2} + \frac{m_c}{h_a^2}$$

$$= \frac{1}{4r^2s^2}\cdot\left(b^2m_a + c^2m_b + a^2m_c\right) \overset{\text{CBS}}{\le} \frac{1}{4r^2s^2}\cdot\sqrt{\sum_{\text{cyc}} a^4}\cdot\sqrt{\sum_{\text{cyc}} m_a^2} \overset{\text{Mitrinovic}}{\le}$$

$$\frac{1}{4r^2.27r^2}\cdot\sqrt{2\sum_{\text{cyc}} a^2b^2 - 16r^2s^2}\cdot\sqrt{\frac{3}{4}\sum_{\text{cyc}} a^2} \overset{\substack{\text{Goldstone}\\ \text{and}\\ \text{Leibnitz}}}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2-16r^2)s^2}\cdot\sqrt{\frac{3}{4}.9R^2}$$

$$\overset{\text{Mitrinovic}}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2-16r^2).\frac{27R^2}{4}}\cdot\sqrt{\frac{3}{4}.9R^2} = \frac{R^2.\sqrt{\frac{R^2-2r^2}{2}}}{4r^4}$$

$$\overset{?}{\le} \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right) = \frac{81R^5 - 2560r^5}{32r^6}$$

$$\Leftrightarrow \left(81R^5 - 2560r^5\right)^2 \overset{?}{\ge} 64R^4r^4\left(\frac{R^2-2r^2}{2}\right)$$

$$\Leftrightarrow 6561t^{10} - 32t^6 - 414720t^5 + 64t^4 + 6553600 \overset{?}{\ge} 0\left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}6561t^8 + 26244t^7 + 78732t^6 + 209952t^5 + 524848t^4\\ +844864t^3 + 1280128t^2 + 1741056t + 1843712\end{pmatrix} +410624\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

$$\text{Again,} \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \ge \frac{h_a}{m_b^2} + \frac{h_b}{m_c^2} + \frac{h_c}{m_a^2} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{\text{cyc}}\frac{1}{m_a}\right)^2}{\sum_{\text{cyc}}\frac{1}{h_a}}$$

$$\overset{\text{Bergstrom}}{\geq} r\left(\frac{9}{\sum_{cyc} m_a}\right)^2 \overset{\text{Leuenberger + Euler}}{\geq} r\left(\frac{9}{\frac{9R}{2}}\right)^2 = \frac{4r}{R^2} = \frac{4r^5}{R^2 r^4} \overset{\text{Euler}}{\geq} \frac{4r^5}{R^2\left(\frac{R}{2}\right)^4} = \frac{64r^5}{R^6}$$

$$\therefore \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \geq \frac{64r^5}{R^6} \text{ and so,}$$

$$\frac{64r^5}{R^6} \leq \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \leq \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

$\forall\, \Delta\, ABC,$ " = " iff $\Delta\, ABC$ is equilateral (QED)