In any Δ ABC, the following relationship holds :

$$\frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \le \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{m_a m_b}{h_a^2 h_b^2} + \frac{m_b m_c}{h_b^2 h_c^2} + \frac{m_c m_a}{h_c^2 h_a^2}$$

$$\overset{\text{Panaitopol}}{\le} \left(\frac{R}{2r}\right)^2 \cdot \sum_{\text{cyc}} \frac{1}{h_b h_c} = \left(\frac{R}{2r}\right)^2 \cdot \sum_{\text{cyc}} \frac{4R^2.bc}{ca.ab.bc} \le \left(\frac{R}{2r}\right)^2 \cdot \frac{4R^2}{16R^2r^2s^2} \cdot \sum_{\text{cyc}} a^2$$

$$\overset{\substack{\text{Leibnitz} \\ \text{and} \\ \text{Mitrinovic}}}{\le} \frac{R^2.9R^2}{16r^4.27r^2} \therefore \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{R^4}{48r^6} \to (1)$$

$$\text{Again, } \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right) \overset{\text{Euler}}{\ge} \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}.64.\left(\frac{R}{r}\right)^4 - 19682\right)$$

$$= \frac{19683R^4 - 19682.16r^4}{48r^6} \overset{?}{\ge} \frac{R^4}{48r^6} \Leftrightarrow 19682(R^4 - 16r^4) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right) \ge \frac{R^4}{48r^6} \overset{\text{via (1)}}{\ge} \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2}$$

$$\Rightarrow \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right)$$

$$\text{Also, } \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \ge \frac{h_a h_b}{m_a^2 m_b^2} + \frac{h_b h_c}{m_b^2 m_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \overset{\text{Panaitopol}}{\ge}$$

$$\frac{4r^2}{R^2}\cdot\sum_{\text{cyc}}\frac{1}{m_b m_c} \overset{\text{Bergstrom}}{\ge} \frac{4r^2}{R^2}\cdot\frac{9}{\sum_{\text{cyc}} m_b m_c} \ge \frac{4r^2}{R^2}\cdot\frac{9}{\sum_{\text{cyc}} m_a^2} = \frac{4r^2}{R^2}\cdot\frac{9}{\frac{3}{4}\cdot\sum_{\text{cyc}} a^2}$$

$$\overset{\text{Euler}}{\ge} \frac{4r^2}{R^2}\cdot\frac{9}{\frac{3}{4}.9R^2} = \frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^2 \overset{\text{Euler}}{\ge} \frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^2\cdot\left(\frac{2r}{R}\right)^8$$

$$\therefore \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \ge \frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \textit{ and so,}$$

$$\frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \le \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right)$$

$\forall\, \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)