ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{m_a^3}{h_b}+\frac{m_b^3}{h_c}+\frac{m_c^3}{h_a}\overset{Panaitopol}{\le}$$

$$\left(\frac{\frac{Rs}{a}}{\frac{2rs}{b}}\right)\left(\frac{2b^2+2c^2+2a^2-3a^2}{4}\right)+\left(\frac{\frac{Rs}{b}}{\frac{2rs}{c}}\right)\left(\frac{2c^2+2a^2+2b^2-3b^2}{4}\right)$$

$$+\left(\frac{\frac{Rs}{c}}{\frac{2rs}{a}}\right)\left(\frac{2a^2+2b^2+2c^2-3c^2}{4}\right)=\frac{R}{2r}\left(\left(\frac{\sum_{cyc}a^2}{2}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\frac{3}{4}\sum_{cyc}ab\right)$$

$$\overset{Leibnitz}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\frac{3R}{8r}.4Rrs.\sum_{cyc}\frac{1}{a}\overset{\substack{CBS\\and\\Bergstrom}}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\sqrt{\sum_{cyc}a^2\cdot\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}$$

$$-\frac{3R}{8r}.4Rrs.\frac{9}{2s}\overset{\substack{Leibnitz\\and\\Goldstone}}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\sqrt{9R^2.\frac{4R^2s^2}{16R^2r^2s^2}}-\frac{3R}{8r}.4Rrs.\frac{9}{2s}=\frac{27R^4}{8r^2}-\frac{27R^2}{4}$$

$$\overset{?}{\le}\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}\Leftrightarrow 81R^5-4R^4r+8R^2r^3-2560r^5\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(81R^4+158R^3r+316R^2r^2+640Rr^3+1280r^4)\overset{?}{\ge}0\to\text{true}$$

$$\because R\overset{Euler}{\ge}2r\therefore\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since

$h_a\le w_a\le m_a$ (and analogs), and by using Rearrangement inequality, we have

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}=\frac{h_cm_a^4+m_aw_b^4+w_bh_c^4}{m_aw_bh_c}\le\frac{m_cm_a^4+m_am_b^4+m_bm_c^4}{h_ah_bh_c}$$

$$\le\frac{m_a^5+m_b^5+m_c^5}{h_ah_bh_c}.$$

Also, we have

$$\bullet\ h_a h_b h_c = \frac{2s^2r^2}{R} \overset{Cosnita-Turtoiu}{\geq} 27r^3 \ \ (1)$$

$$\bullet\ m_a{}^5 + m_b{}^5 + m_c{}^5 = \left(\sum_{cyc} m_a\right)^5 - 5\left(\sum_{cyc} m_a{}^2 + \sum_{cyc} m_b m_c\right)\prod_{cyc}(m_b + m_c)$$

$$\overset{Leuenberger\ \&\ AM-GM}{\leq} (4R + r)^5 - 5.6\sqrt[3]{(m_a m_b m_c)^2}.8m_a m_b m_c$$

$$\overset{Euler\ \&\ m_a \geq h_a}{\leq} \left(\frac{9R}{2}\right)^5 - 240\sqrt[3]{(h_a h_b h_c)^5} \overset{(1)}{\leq} \left(\frac{9R}{2}\right)^5 - 240(3r)^5$$

$$= \frac{27^2(81R^5 - 2560r^5)}{32}.$$

Using these results, we obtain

$$\frac{m_a{}^3}{w_b} + \frac{w_b{}^3}{h_c} + \frac{h_c{}^3}{m_a} < \frac{27}{32}.\frac{81R^5 - 2560r^5}{r^3}.$$

Equality holds iff ΔABC is equilateral.