ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$27r^2.\left(\frac{2r}{R}\right)^6 \le \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6-104r^2\right)$$

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$$\frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \le \frac{m_a^2m_b^2}{h_ah_b}+\frac{m_b^2m_c^2}{h_bh_c}+\frac{m_c^2m_a^2}{h_ch_a} \overset{Panaitopol}{\le}$$

$$\left(\frac{R}{2r}\right)^2\sum_{cyc} m_bm_c \le \frac{R^2}{4r^2}.\sum_{cyc} m_a^2 \overset{Panaitopol}{\le} \frac{R^2}{4r^2}.\frac{3}{4}.9R^2$$

$$\therefore \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \le \frac{27R^4}{16r^2} \to (1)$$

$$\text{Again, } \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6-104r^2\right) \overset{Euler}{\ge} \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^2-104r^2\right)$$

$$= \frac{27(27R^4-416r^4)}{16r^2} \overset{?}{\ge} \frac{27R^4}{16r^2} \Leftrightarrow 26(R^4-16r^4) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r$$

$$\therefore \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6-104r^2\right) \ge \frac{27R^4}{16r^2} \overset{via\ (1)}{\ge} \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a}$$

$$\therefore \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6-104r^2\right)$$

$$\text{Also, } \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \ge \frac{h_a^2h_b^2}{m_am_b}+\frac{h_b^2h_c^2}{m_bm_c}+\frac{h_c^2h_a^2}{m_cm_a} \overset{Panaitopol}{\ge} \frac{4r^2}{R^2}.\sum_{cyc} h_bh_c$$

$$= \frac{4r^2}{R^2}.\sum_{cyc}\frac{ca.ab}{4R^2} = \frac{4r^2.4Rrs.2s}{R^2.4R^2} = \frac{8r^3}{R^3}.s^2 \overset{Mitrinovic}{\ge} \left(\frac{2r}{R}\right)^3.27r^2 \overset{Euler}{\ge}$$

$$\left(\frac{2r}{R}\right)^3.27r^2.\left(\frac{2r}{R}\right)^3 \Rightarrow \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \ge 27r^2.\left(\frac{2r}{R}\right)^6 \text{ and so,}$$

$$27r^2.\left(\frac{2r}{R}\right)^6 \le \frac{m_a^2m_b^2}{w_aw_b}+\frac{w_b^2w_c^2}{h_bh_c}+\frac{h_c^2h_a^2}{m_cm_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6-104r^2\right)$$

$\forall\ \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)