ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le$$

$$\frac{m_a+m_b}{(h_a+h_b)^2}+\frac{m_b+m_c}{(h_b+h_c)^2}+\frac{m_c+m_a}{(h_c+h_a)^2}\overset{\text{Panaitopol}}{\le}\frac{R}{2r}\cdot\sum_{\text{cyc}}\frac{h_b+h_c}{(h_b+h_c)^2}$$

$$=\frac{R}{2r}\cdot 2R\sum_{\text{cyc}}\frac{1}{ca+ab}\overset{\text{A-G}}{\le}\frac{R^2}{2r}\cdot\sum_{\text{cyc}}\frac{1}{a\sqrt{bc}}=\frac{R^2}{2r}\cdot\sum_{\text{cyc}}\left(\sqrt{\frac{1}{ab}}\cdot\sqrt{\frac{1}{ac}}\right)$$

$$\overset{\text{CBS}}{\le}\frac{R^2}{2r}\cdot\sqrt{\sum_{\text{cyc}}\frac{1}{ab}}\cdot\sqrt{\sum_{\text{cyc}}\frac{1}{ab}}=\frac{R^2}{2r}\cdot\frac{2s}{4Rrs}$$

$$\therefore\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le\frac{R}{4r^2}\to(1)$$

$$\text{Again, }\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\overset{\text{Euler}}{\ge}\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot 16\left(\frac{R}{r}\right)-80\right)$$

$$\Rightarrow\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\ge\frac{81R-160r}{4r^2}=\frac{80(R-2r)}{4r^2}+\frac{R}{4r^2}\overset{\text{Euler}}{\ge}\frac{R}{4r^2}$$

$$\overset{\text{via }(1)}{\ge}\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}$$

$$\therefore\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}$$

$$\le\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\ \forall\,\Delta\,\text{ABC},''=''\ \text{iff }\Delta\,\text{ABC is equilateral (QED)}$$