ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, the following relationship holds :

$$\frac{8r^2}{R^3} \le \frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{R^2}{4r^3} \text{ and}$$

$$\frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^4 \le \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{1}{r^2}\cdot\left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} = \frac{4r^2}{4r^3} \overset{\text{Euler}}{\le} \frac{R^2}{4r^3} \text{ and}$$

$$\frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \ge \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leuenberger + Euler}}}{\ge} \frac{9}{\frac{9R}{2}} = \frac{2.4r^2}{R.4r^2} \overset{\text{Euler}}{\ge} \frac{8r^2}{R^3}$$

$$\frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \sum_{\text{cyc}} \frac{1}{h_a^2} = \frac{\sum_{\text{cyc}} a^2}{4r^2s^2} \overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le} \frac{9R^2}{4r^2.27r^2}$$

$$\Rightarrow \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{R^2}{12r^4} \to (1) \text{ and } \frac{1}{r^2}.\left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right) \overset{\text{Euler}}{\ge} \frac{3R^2 - 8r^2}{12r^4}$$

$$\overset{\text{Euler}}{\ge} \frac{3R^2 - 2R^2}{12r^4} \overset{\text{via (1)}}{\ge} \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \text{ and, } \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \ge \sum_{\text{cyc}} \frac{1}{m_a^2} \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\ge}$$

$$\frac{9}{\frac{3}{4}.9R^2} = \frac{4}{3R^2} \overset{\text{Euler}}{\ge} \frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^4 \therefore \frac{8r^2}{R^3} \le \frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{R^2}{4r^3} \text{ and}$$

$$\frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^4 \le \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{1}{r^2}\cdot\left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right)$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)