ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}\le\frac{1}{4r}\cdot\left(27\cdot\left(\frac{R}{2r}\right)^8-26\right)$$

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$$\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}\le\sum_{cyc}\frac{m_b^2+m_c^2}{(h_b+h_c)^3}$$

$$\le\sum_{cyc}\frac{(h_b+h_c)(m_b^2+m_c^2)}{8h_bh_c(h_b^2+h_c^2)}$$

$$\left(\begin{array}{c}\because\forall x,y>0,(x+y)^4=\left((x^2+y^2)+2xy\right)^2\overset{A-G}{\ge}8xy(x^2+y^2)\\ \Rightarrow(x+y)^3\ge\frac{8xy(x^2+y^2)}{x+y}\end{array}\right)$$

$$\overset{Panaitopol}{\le}\sum_{cyc}\frac{(h_b+h_c)\cdot\frac{R^2}{4r^2}(h_b^2+h_c^2)}{8h_bh_c(h_b^2+h_c^2)}=\frac{R^2}{32r^2}\cdot\sum_{cyc}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{R^2}{32r^2}\cdot\frac{2}{r}=\frac{R^2}{16r^3}$$

$$and\ \frac{1}{4r}\cdot\left(27\cdot\left(\frac{R}{2r}\right)^8-26\right)\overset{Euler}{\ge}\frac{1}{4r}\cdot\left(27\cdot\left(\frac{R}{2r}\right)^2-26\right)=\frac{27R^2-26\cdot4r^2}{16r^3}$$

$$\therefore\text{ it suffices to prove : }\frac{27R^2-26\cdot4r^2}{16r^3}\ge\frac{R^2}{16r^3}\Leftrightarrow26(R^2-4r^2)\ge0$$

$$\to\text{ true via Euler }\therefore\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}$$

$$\le\frac{1}{4r}\cdot\left(27\cdot\left(\frac{R}{2r}\right)^8-26\right)\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$