ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ n, m ≥ 0, the following relationships holds :

$$\frac{r_a^n}{m_a^m}+\frac{r_b^n}{w_b^m}+\frac{r_c^n}{h_c^m}\geq\frac{3^{n-m+1}.2^m.r^n}{R^m}\ \textit{and}\ \frac{m_a^n}{r_a^m}+\frac{w_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\geq\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

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$$\frac{r_a^n}{m_a^m}+\frac{r_b^n}{w_b^m}+\frac{r_c^n}{h_c^m}\geq\frac{r_a^n}{m_a^m}+\frac{r_b^n}{m_b^m}+\frac{r_c^n}{m_c^m}\overset{A-G}{\geq}3.\sqrt[3]{\frac{\left(\prod_{cyc}r_a\right)^n}{\left(\prod_{cyc}m_a\right)^m}}\overset{m_am_bm_c\leq\frac{Rs^2}{2}}{\geq}$$

$$3.\sqrt[3]{\frac{(rs^2)^n}{\left(\frac{Rs^2}{2}\right)^m}}\overset{\text{Mitrinovic}}{\geq}3.\sqrt[3]{\frac{(r.27r^2)^n}{\left(\frac{R}{2}.\frac{27R^2}{4}\right)^m}}=\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

$$\forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral}$$

$$\frac{m_a^n}{r_a^m}+\frac{w_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\geq\frac{h_a^n}{r_a^m}+\frac{h_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\overset{A-G}{\geq}3.\sqrt[3]{\frac{\left(\prod_{cyc}h_a\right)^n}{\left(\prod_{cyc}r_a\right)^m}}=3.\sqrt[3]{\frac{\left(\frac{2r^2s^2}{R}\right)^n}{(rs^2)^m}}$$

$$\overset{\substack{\text{Gerretsen + Euler}\\ \textit{and}\\ \text{Mitrinovic + Euler}}}{\geq}3.\sqrt[3]{\frac{\left(\frac{r^2.27Rr}{R}\right)^n}{\left(\frac{R}{2}.\frac{27R^2}{4}\right)^m}}=\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

$$\forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral}$$

$$\boxed{\text{Proof of } m_am_bm_c\leq\frac{Rs^2}{2}}$$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\text{Now,} \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \rightarrow (2)$$

$$\text{Also,} \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) =$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \rightarrow (3) \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(\begin{array}{c} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{array}\right)$$

$$= \frac{1}{16}\left(s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\right)$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \overset{(\bullet)}{\le} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R + r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \underbrace{\overset{?}{\ge}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(a), (b) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$$

$$\ge 20rs^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \ge 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \overset{(\bullet\bullet\bullet)}{\ge} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(c)} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(d)} 27r^2(4R^2 + 4Rr + 3r^2)$$

$(c), (d) \Rightarrow$ **in order to prove** $(\bullet\bullet\bullet)$, **it suffices to prove :**

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \ge 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \ge 0 \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\big((t - 2)(224t + 309) + 648\big) \ge 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \le \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \le \frac{Rs^2}{2} \text{ (QED)}$$